Truth Table Evaluation of Statement Forms

The truth conditions for truth functional statements or statement forms can be evaluated by means of truth tables. A truth table gives an exhaustive list of all possible combinations of truth values for a given statement form. Each row of the table represents one set of possible combinations of truth values.

The number of rows in a table is 2^n where n= the number of variables in the statement form or number of constants in the statement. Thus a statement form with one variable will have two rows, a statement form with two variables will have four rows, a statement with three variables will have eight rows, etc.

The column for the first variable will have the first half T's and the second half F's. The column for the second variable will have the first quarter T's, the second quarter F's, the third quarter T's, and the forth quarter F's. If there are three variables, then the third column will be divided into eighths. This pattern is followed so the column under the last variable alternates T's and F's. The variable columns for a three variable statement form are as follows.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

In addition to the columns for the variables, there will be columns for each truth functional operator in the statement form. These columns are calculated from the columns for the variables, beginning with the least complex component forms and working toward the more complex forms.

In order to illustrate the above procedure, we will work out the examples in steps, and number the columns in the order that they are calculated. We will begin with the example, ~[(p ∨ q) ⊃ (p ≡ q)]. The first step is to set up the table as below.

```
 p  q|  ~[(p ∨ q) ⊃ (p ≡ q)]
 T  T|
 T  F|
 F  T|
 F  F|
```

The columns for the variables are on the left and the statement form is on the right. The main operator for the statement form is the negation in front of the left bracket. The column for it will be calculated last. Within the brackets, the main operator is the horseshoe. It will be the next to last column calculated. We can begin by calculating the column for the vee.

```
                          1
 p  q|  ~[(p ∨ q) ⊃ (p ≡ q)]
 T  T|           T
 T  F|           T
 F  T|           T
 F  F|           F
```

When both disjuncts are true, the disjunction is true (row 1). When the first disjunct is true and the second is false, the disjunction is true (row 2). When the first disjunct is false and the second is true, the disjunction is true (row three). When both disjuncts are false, the disjunction is false (row 4).

Next we calculate the column for the biconditional.

```
                          1         2
 p  q|  ~[(p ∨ q) ⊃ (p ≡ q)]
 T  T|           T         T
 T  F|           T         F
 F  T|           T         F
 F  F|           F         T
```

When both terms of the biconditional are true, the biconditional is true (row 1). When one term is true and the other term is false, the biconditional is false (rows 2 and 3). When both terms are false, the biconditional is true (row 4).

The third step is to calculate the column for the conditional. In doing so, we will be using column 1 (the antecedent) and column 2 (the consequent).

p	q	~[(p ∨ q) ⊃ (p ≡ q)]		
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

When the antecedent is true and the consequent is true, the conditional is true (row 1). When the antecedent is true and the consequent is false, the conditional is false (rows 2 and 3). When the antecedent is false and the consequent is true, the conditional is true (row 4).

The last step is the column for the negation. For this, we use column 3.

		4	1	3	2
p	q	~[(p ∨ q) ⊃ (p ≡ q)]			
T	T	F	T	T	T
T	F	T	T	F	F
F	T	T	T	F	F
F	F	F	F	T	T

Column 4 is the table for the statement form. If true statements are substituted for p and q, the statement will be false (row 1). If a true statement is substituted for one variable and a false statement is substituted for the other variable, the statement will be true (rows 2 and 3). If false statements are substituted for p and q, the statement will be false.

We will work through one more example step by step.

		1
p	q	(p ∨ ~q) ⊃ (p & q)
T	T	F
T	F	T
F	T	F
F	F	T

The first step is the negation of q. Next we do the disjunction of p and ~q.

```
              2 1
   p  q │ (p ∨ ~q) ⊃ (p & q)
   T  T │    T F
   T  F │    T T
   F  T │    F F
   F  F │    T T
```

After completing column 2, we drew a line through column 1. This is because we will no longer have to refer to that column, and drawing the line through it makes it visually easier to avoid mistakes.
      The next column is for the conjunction of p and q.

```
              2 1         3
   p  q │ (p ∨ ~q) ⊃ (p & q)
   T  T │    T F         T
   T  F │    T T         F
   F  T │    F F         F
   F  F │    T T         F
```

 The last column is for the conditional. It is calculated using columns 2 and 3.

```
              2 1         3
   p  q │ (p ∨ ~q) ⊃ (p & q)
   T  T │    T F    T    T
   T  F │    T T    F    F
   F  T │    F F    T    F
   F  F │    T T    F    F
```

Once column 4 is completed, we draw lines through columns 2 and 3 since we have finished with them. We are left with the column for the main operator.

Exercise 4.1

 Construct truth tables for the following statement forms.

1. p ⊃ (p & q)
2. p ⊃ (p ∨ q)
3. ~p ∨ (p & ~q)
4. p & ~(p & q)
5. (p & q) ≡ (p ∨ q)

6. $(p \lor q) \supset p$
7. $\sim[p \lor (p \,\&\, q)]$
8. $p \supset (q \supset r)$
9. $(p \,\&\, q) \supset (p \lor r)$
10. $\sim[p \equiv (p \,\&\, q)] \supset r$

Tautologies, Contradictions, and Contingencies

Truth table evaluations of statement forms reveal certain statements to be true in virtue of their form alone. Such statements are called tautologies. A tautology is a substitution instance of a tautologous statement form. The form, "$p \supset (p \lor q)$," is one such form, as is shown in the following table.

p q	p $\supset$ (p $\lor$ q)	
T T	T	T
T F	T	T
F T	T	T
F F	T	F

The truth conditions for the statement form are given in the column under the horseshoe. No matter what the truth values of statements substituted for the variables p and q might be, the resulting statement will be true.

Certain other statements are false in virtue of their form alone. These are contradictions. The form, "$(p \,\&\, q) \equiv (\sim p \lor \sim q)$," is a contradiction.

p q	(p & q) $\equiv$ ($\sim$p $\lor$ $\sim$q)				
T T	T	F F	F F		
T F	F	F F	T T		
F T	F	F T	T F		
F F	F	F T	T T		

The truth conditions for the statement form are given in the column under the triple bar. No matter what the truth values of the statements substituted for the variables p and q might be, the resulting statement will be false.

Statements forms whose truth value varies with the truth or falsity of the statements substituted for the variables in the form are contingencies. The form, "$(p \,\&\, q) \supset (\sim p \lor \sim q)$," is contingent.

```
p  q │ (p & q) ⊃ (~p ∨ ~q)
T  T │    T    F F   F F
T  F │    F    T F   T T
F  T │    F    T T   T F
F  F │    T    T T   T T
```

The column under the horseshoe defines the truth conditions for the
statement form. If true statements are substituted for the variables, the
resulting statement is true. If a false statement is substituted for one or both
of the variables, the resulting statement will be false.

Exercise 4.2

 Use truth tables to determine if the following statement forms are tautologous,
 contradictory, or contingent.

1. p ⊃ [p ∨ (p & q)]
2. p ⊃ (p & q)
3. ~p ∨ (p ∨ q)
4. p ⊃ (q ∨ ~q)
5. ~[(p & ~p) ⊃ q]
6. (p ⊃ q) ≡ (~p ∨ q)
7. [p & (p ⊃ q)] ≡ (p & ~q)
8. [p ⊃ (q & r)] ⊃ (p ⊃ r)
9. ~[(p & r) ⊃ (q ⊃ r)]
10. ~[(p ∨ r) ⊃ (q ⊃ r)]

Logical Equivalence and Logical Implication

        Earlier we defined material equivalence and material implication by means of
truth tables. Logical equivalence and logical implication are stronger relations. Two
statement forms are logically equivalent if and only if the statement of their material
equivalence is a tautology. The standard symbol for logical equivalence is a double
arrow, "⇔".
        As an example, we will use truth tables to demonstrate the truth of two logical
equivalences known as DeMorgan's Rules. They are:

~(p ∨ q) ⇔ (~p & ~q)
~(p & q) ⇔ (~p V ~q)

Here is the table for the first equivalence.

```
p q | ~(p ∨ q)  ≡  (~p & ~q)
T T | F   T      T   F  F   F
T F | F   T      T   F  F   T
F T | F   T      T   T  T   F
F F | T   F      T   T  T   T
```

Since the material equivalence is a tautology, "~(p ∨ q) ⇔ (~p & ~q)" is true.

Here is the table for ~(p & q) ⇔ (~p ∨ ~q).

```
p q | ~(p & q)  ≡  (~p ∨ ~q)
T T | F   T   T   T   F F F
T F | T   F   T   T   F T T
F T | T   F   T   T   T T F
F F | T   F   T   T   T T T
```

Since "~(p & q) ≡ (~p ∨ ~q)" is a tautology, "~(p & q) ⇔ (~p ∨ ~q)" is true.

One statement form logically implies another if the statement of their material implication is a tautology. The symbol for logical implication is "⇒." The following table demonstrates the truth of [(p & q) ⇒ q)].

```
p   q | (p & q) ⊃ q
T   T |   T     T
T   F |   F     T
F   T |   F     T
F   F |   F     T
```

Exercise 4.3

Use truth tables to demonstrate the following logical equivalences.

1. p ⇔ ~~p   2. [(p & q) ⊃ r] ⇔ [p ⊃ (q ⊃ r)]
3. (p ⊃ q) ⇔ (~q ⊃ ~p)   4. (p ⊃ q) ⇔ (~p ∨ q)

Use truth tables to demonstrate the following logical implications.

5. [(p ⊃ q) & p] ⇒ q   6. [(p ⊃ q) & ~ q] ⇒ ~p
7. [(p ∨ q) & ~q] ⇒ p   8. p ⇒ (p ∨ q)