TRUTH FUNCTIONAL STATEMENTS

In recognizing that a sentence expresses a statement, it is not necessary to know whether the statement expressed is true or false, only that it is one or the other. This property that allows us to recognize something as a statement without being able to determine its specific truth value is *bivalence*. It is the property of having to have one of two values, in this case, either the value true or the value false.

A compound statement is a statement that has some other statement or statements as a component of it. A simple statement is one that has no other statement as a component. The sentence

> Jack went up the hill and Jill followed after.

expresses a compound statement. The statement has as components

- Jack went up the hill.
- Jill followed after.

The sentence

> Jack went up the hill.

expresses a simple statement. It has no component that expresses a statement.

A truth functionally compound statement is a statement whose truth or falsity is a function of the truth or falsity of one or more component statements. A truth functionally simple statement is one whose truth or falsity is not a function of a component statement. The statement expressed by

> Jack went up the hill and Jill went up the hill.

is truth functionally compound. If either *Jack went up the hill* or *Jill went up the hill* are false, then the compound statement is false. Only if both component statements are true is the truth functionally compound statement true.

English offers a variety of ways to express compound statements. The compound sentence is only one of these ways. You can have a compound subject, as in

> Jack and Jill went up the hill.

where each of the logical subjects receives the predicate. Thus the above sentence expresses a statement with the components expressed by

- Jack went up the hill.
- and
Jill went up the hill.

You can also have a compound predicate, as in

Jack went up the hill and fetched a pail of water.

This sentence expresses a statement with the components

Jack went up the hill.

and

Jack fetched a pail of water.

Grammatically complex sentences can also be used to express truth functionally compound statements. The sentence

If wishes are horses, then beggars ride.

is truth functional in that its truth or falsity is dependent upon the truth or falsity of

Wishes are horses.

and

Beggars ride.

Even a grammatically simple sentence can express a truth functionally compound statement. The sentence

Jack did not fetch water.

expresses a truth functionally compound statement whose truth or falsity is a function of the truth or falsity of the statement expressed by

Jack fetched water.

If the component is true, then the compound is false. If the component is false, then the compound is true.

Exercise 3.1

Which of the following sentences express truth functionally compound statements?

1. Roses are red and violets are blue.
2. Neither roses nor violets grow in the desert.
3. All roses are flowers.
4. John said he had seen black roses.
5. If roses get mildew, then they will not flourish.
6. Augustus will receive either roses or violets for his birthday.
7. Some roses are not red, but others are.
8. John enjoys roses, yet he does not grow them.
9. While Susan hates roses, she still works in a flower shop.
10. It is false that all roses are red.

STATEMENT VARIABLES AND STATEMENT CONSTANTS

A statement variable is a placeholder, a symbol for which a statement might be substituted. For statement variables we will use the lower case letters of the alphabet, beginning with "p."

A statement constant is an abbreviation of a statement. We will use capital letters for statement constants. A constant always stands for some particular statement, while a variable is a placeholder into which a statement may be substituted.

The statement

Athens and Sparta are Greek cities.

can be abbreviated as

A and S.

The "A" abbreviates "Athens is a Greek city" and the "S" abbreviates "Sparta is a Greek city."

"A and S" is a substitution instance of "p and q." That is, if we replace the variable "p" with the constant "A" and the variable "q" with the constant "S," then from "p and q" we get "A and S." When a statement is a substitution instance of a given statement form, then the statement has that form. A statement S has a statement form F if and only if S is the result of replacing the variables in F with statements, the same statement replacing each occurrence of a given variable in F.

A statement may have more than one form. This is the case when truth functionally compound statements are substituted for the variables in a statement form. For example, the statement

Athens and Sparta are Greek city states, but Rome is not.

results from substituting "Athens and Sparta are Greek city states" for "p" and "Rome is not a Greek city state" for "q" in "p and q." It also results from substituting "Athens is a Greek city state" for "p," "Sparta is a Greek city state" for "q," and "Rome is not a Greek city state" for "r" in the form, "p and q and r."

Because a statement may have more than one form, it is sometimes useful to distinguish between a form and the specific form of a given statement. A statement S is a
specific substitution instance of a statement form $F$ if and only if $S$ is the result of replacing the variables in $F$ with truth functionally simple statements, the same statement replacing each occurrence of a given variable.

Any statement that is substituted into an expression containing “p” will either be true or it will be false. This is also the case for statements substituted for "q," "r," or any other variable. If an expression contains two variables, "p" and "q," there are four possible combinations of truth values. These can be pictured in a table.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Statements substituted for "p" and "q" could both be true. The statement substituted for "p" could be true and the statement substituted for "q" could be false. The statement substituted for "p" could be false and the statement substituted for "q" could be true. Finally, statements substituted for "p" and "q" could both be false. There are no other possible combinations for an expression containing exactly two variables.

**NEGATION, CONJUNCTION, AND DISJUNCTION**

In addition to statement variables, we want symbols for the terms that operate upon statements and are used to combine statements with other statements. The first of these that we will introduce is negation. For any statement "p," if "p" is true, its negation is false. If "p" is false, its negation is true.

The symbol we will use for negation is the tilde, "~." This symbol is defined by the following table.

<table>
<thead>
<tr>
<th>p</th>
<th>~p</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

If "p" is true, then "~p" is false; if "p" is false, then "~p" is true. Negation operates on one and only one expression.

As we have already seen, a variety of English sentences may express the same statement. The denial of "Jack went up the hill" may be expressed as:

Jack did not go up the hill.
It is not the case that Jack went up the hill.
It is false that Jack went up the hill.

Any of these would be symbolized as "~J."

A conjunction is true if and only if both of the component statements (called conjuncts) are true. If one, or both, of the conjuncts are false, the conjunction is false.
The symbol we use for conjunction is the ampersand, "&." The table defining conjunction is:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p &amp; q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

If "p" and "q" are both true, "p & q" is true. If "p" is true and "q" is false, "p & q" is false. If "p" is false and "q" is true, "p & q" is false. If "p" and "q" are both false, then "p & q" is false.

Any of the following sentences would be symbolized "G & M."

- George and Martha are over the hill.
- George is over the hill and Martha is over the hill.
- George is over the hill and so is Martha.
- George is over the hill but so is Martha.

The English "or," used to express disjunction, is ambiguous. It may be used inclusively, "one, the other, or both," or exclusively, "one, the other, but not both." We will introduce a symbol only for the inclusive sense. Exclusive disjunction is used less frequently than the inclusive, and can be handled without a special symbol.

A disjunction is true if one, or both, of the component statements (called disjuncts) is true. A disjunction is false only if both of the disjuncts are false. Disjunction is symbolized by a vee or wedge, "\(\lor\)," and is defined by the following table.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p (\lor) q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

If "p" and "q" are both true, then "p \(\lor\) q" is true. If "p" is true and "q" is false, "p \(\lor\) q" is true. If "p" is false and "q" is true, "p \(\lor\) q" is true. Only if "p" and "q" are both false is "p \(\lor\) q" false.
Exercise 3.2

Symbolize the statements expressed by each of the following sentences. Use the suggested notation.

1. March is windy, but October is dry. (M, O)
2. January and February are cold. (J, F)
3. July and August are hot. (J, A)
4. Either May or June is pleasant for picnicking. (M, J)
5. April is wet, but not September. (A, S)
6. December and January are not warm months. (D, J)
7. July is not cool, but March is. (J, M)
8. May is pleasant, yet June is hot (J, M)
9. John's birthday is in May or June. (M, J)
10. Jane's birthday is in February, or she is not Aquarian. (F, A)

CONDITIONALS AND BICONDITIONALS

Conditionals are most commonly expressed in English in "If...then..." sentences. The clause following the "if" is called the antecedent of the conditional. The clause following the "then" is called the consequent. The logical relation between antecedent and consequent in a conditional statement is material implication. The symbol for material implication is a right facing horseshoe, $\Rightarrow$. The antecedent is on the left side of the horseshoe and the consequent is on the right side.

Below is the table for material implication.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p (\Rightarrow) q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
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<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

If the antecedent and consequent are both true, the conditional is true. If the antecedent is true and the consequent is false, the conditional is false. If the antecedent is false and the consequent is true, the conditional is true. If antecedent and consequent are both false, the conditional is true.

While "If...then..." is the most common English form for expressing conditionals, it is not the only form. The "then" is usually optional, and is often left out. The order of the antecedent and consequent can be reversed. "When" can be used instead of "if." And "only if" can be used to designate the consequent. All of the following forms are equivalent.

If p, then q.
If \( p \), \( q \).
\( q \), if \( p \).
\( p \) only if \( q \).
When \( p \), then \( q \).
When \( p \), \( q \).

Each of the above is symbolized "\( p \supset q \)."

Conditional statements can be used to indicate necessary or sufficient conditions.

From the table,

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \supset q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
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<tr>
<td>F</td>
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<td>T</td>
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</tbody>
</table>

we can see that if the conditional is true, and the antecedent is true, the consequent must be true. Thus the antecedent is sufficient for the consequent.

If the consequent is false and the statement is true, as in line 4,

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \supset q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
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<td>F</td>
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</tr>
</tbody>
</table>

then the antecedent is false. Thus the consequent is necessary for the antecedent.

An expression of the form "if and only if" is a biconditional. The logical relation expressed is material equivalence. Material equivalence is symbolized by "\( \equiv \)," a triple bar. The table for material equivalence is

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \equiv q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>F</td>
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<td>T</td>
</tr>
</tbody>
</table>

A statement of material equivalence is true if both component statements have the same truth value. It is false if the component statements have different truth values.

**Exercise 3.3**

Symbolize the statements expressed by each of the following sentences. Use the
GROUPING

As truth functionally compound statements increase in complexity, it is necessary to develop a means to avoid ambiguity. For example, the expression

$$A \land B \lor C$$

could be read "A, and B or C," or it could be read "A and B, or C." How the expression is read makes a difference in the truth conditions, and thus in the statement expressed. The first reading,

$$A, \text{ and } B \text{ or } C$$

to be true, requires that \(A\) be true, and that either \(B\) or \(C\) be true. The truth of \(A\) is necessary. The second reading,

$$A \text{ and } B, \text{ or } C$$

makes \(C\) sufficient for the truth of the statement. \(A\) need not be true.
To avoid ambiguity, we will use parentheses, "(...)," brackets, "[..]," and braces, "{..}," for punctuation. We will stipulate that negation applies to the smallest unit allowed for by the punctuation. A disjunction will have exactly two disjuncts and a conjunction will have exactly two conjuncts.

The statement

John is not a senior and Harry is.

is symbolized

\( \neg J \land H \).

The statement, "John is a senior," is denied, but not the statement, "Harry is a senior." The statement

It is false that John is a senior and Harry is a senior.

is symbolized

\( \neg (J \land H) \)

for in this case it is the conjunction of the two statements that is denied. Indeed, the statement is more clearly expressed in English by

It is false that Harry and John are both seniors.
Exercise 3.4

Symbolize the statements expressed by each of the following sentences. Use the suggested notation.

1. If Abe and Bill both come to the party then Carol will have a wonderful time. (A, B, C)
2. Neither Abe nor Bill will come to the party. (A, B)
3. If Abe comes to the party, and if Bill does not, then Carol will have a wonderful time. (A, B, C)
4. If Abe does not come to the party, and if Bill does come, then Carol will have a wonderful time. (A, B, C)
5. If neither Abe nor Bill come to the party, then Carol will have a wonderful time. (A, B, C)
6. Either Abe and Carol come to the party, or Bill and Carol come to the party. (A, B, C)
7. If Abe comes to the party, then Bill comes to the party; and if Bill comes to the party, then Carol comes to the party. (A, B, C)
8. Either Abe and Bill come to the party, or, if Carol comes to the party, then Don comes to the party. (A, B, C, D)
9. If and only if Abe comes to the party will Bill come to the party; and only if Carol comes to the party will Don come to the party. (A, B, C, D)
10. If Abe comes to the party, and if Bill comes to the party, and if Carol comes to the party, then Don will come to the party. (A, B, C, D)

EVALUATING THE TRUTH OF COMPOUND STATEMENTS

If we know the truth values of the simple statements in a compound statement, then we can determine the truth value of the compound. Given the compound statement

\[(A \land B) \lor (C \land D)\]

and the truth values A= true, B= false, C= false, and D= true, we can tell that the statement is false. The main operator of the compound is the "\lor". The first disjunct is the conjunction, "A \land B". Since "B" is false, and a conjunction is true only if both conjuncts are true, "A \land B" is false. Likewise, the second disjunct, the conjunction "C \land D", is false because "C" is false. Since both disjuncts of the disjunction are false, the statement is false. This analysis is illustrated by the following diagram.
A more complex example is given below. Here the constants have the values A=true, B=true, C=true, D=false, and E=false.

Since "A" and "B" are both true, "A & B" is true. That conjunction is a disjunct with "C" and the disjunction of two true statements is true. That disjunction, "(A & B) ∨ C", is the antecedent of the conditional of which "D" is the consequent. A conditional with a true antecedent and false consequence is false so 

\[ [(A & B) ∨ C] ⊃ D \]

is false. That conditional is one disjunct of the disjunction, of which "E", also false, is the other disjunct. Therefore the statement is false.

**Exercise 3.5**

Given the values A=true, B=false, C=true, D=false, and E=true, determine the truth values of the following compound statements.

1. \((A & B) ∨ C\)
2. \((A ∨ B) & C\)
3. \((A & B) ∨ (C & D)\)
4. \((A ∨ B) & (C ∨ D)\)
5. \((A ∨ B) ≡ (C ∨ D)\)
6. \((A ∨ B) ⊃ C\)
7. \((A ∨ B) ⊃ (C & D)\)
8. \(A ⊃ \{B ⊃ [C ⊃ (D ⊃ E)]\}\)
9. \( \sim\{[A \supset (B \supset C)] \equiv (D \supset E)\} \)
10. \( \sim[A \& (B \& C)] \lor (D \& E) \)