Quantification

In this and subsequent chapters, we will develop a more formal system of dealing with categorical statements, one that will be much more flexible than traditional logic, allow a deeper analysis of predication, and utilize the reduction tree method for testing statements and arguments.

Constants and Predicates

The basic unit of truth functional logic was the statement. Out of statements, statement variables and truth functional connectives, we developed a statement calculus. The units with which we will now be dealing include predicates and the individuals to whom those predicates are applied. Upper case letters will be used as predicate constants. Lower case letters, "a" through "v" will be used as individual constants. Lower case letters, "x" through "z", will be used as individual variables.

A limitation of traditional categorical logic was the lack of an easy and effective way of dealing with individuals. The statement, "Socrates is mortal," was translated into standard form as "All persons identical to Socrates are mortals." Using "s" as an individual constant abbreviating the proper name, "Socrates," and "Mx" abbreviating the predicate "x is mortal," we can express the statement "Socrates is mortal" as:

Ms

substituting the individual constant "s" for the "x". "Socrates is a wise mortal" can be symbolized as the truth functionally compound statement,

Ws & Ms

which we read as "Socrates is wise and Socrates is mortal."

Exercise 14.1

Using the suggested notation, symbolize the statements expressed by the following sentences.

1. Laches was an Athenian general. (l: Laches; Ax: x was an Athenian; Gx: x was a general)
2. Laches was wise and brave. (l: Laches; Wx: x was wise; Bx: x was brave)
3. Both Laches and Demosthenes were brave generals. (l: Laches; d: Demosthenes; Bx: x was brave; Gx: x was a general).
4. Socrates was wise and brave, but not a general. (s: Socrates; Wx: x was wise; Bx: x was brave; Gx: x as a general)
5. If Laches was a general, then he was a brave Athenian. (l: Laches; Gx: x was a general; Bx: x was brave; Ax: x was an Athenian)

Quantifiers

In traditional logic, statements such as "All humans are mortals" were interpreted as relations of inclusion or exclusion between classes. In the predicate calculus, they are treated as general statements about individuals. The symbol "∀" is used for universal quantification and the symbol "∃" is used for existential quantification.

Bound and Free Variables

A variable is said to be bound if it falls within the scope of a quantifier. It is free if it does not fall within the scope of a quantifier. Only if all variables within an expression are bound is that expression complete or well-formed. The expression,

$$Hx$$

is not a well-formed formula. The free occurrence of the variable "x" leaves the expression uninterpretable in terms of truth or falsity.

Generalization

A general statement may be either existential or universal. From the statement, "Socrates is human," symbolized

$$Hs$$

we may validly infer "Something is human." This would be symbolized

$$\exists x\ Hx$$

and literally read as "There exists an x such that x is human."

The scope of a quantifier is interpreted as applying to the smallest possible unit, just as was negation or the tilde (~). If we existentially generalize from "Socrates is human and mortal"

$$Hs \ \&\ Ms$$

to "There exists an x such that x is both human and mortal," we must symbolize the latter as

$$\exists x(Hx \ \&\ Mx)$$

for the expression

$$\exists x\ Hx \ \&\ Mx$$

leaves the second occurrence of the variable "x" unbound. It is useful to compare "There exists an x such that x is both human and mortal" with "There exists an x which is human and there exists an x which is mortal." The latter is symbolized

$$\exists x\ Hx \ \&\ \exists x\ Mx.$$

The above does not assert that the thing which is human is the same thing which is mortal.

The universally general statement "Everything is mortal" is symbolized

$$\forall x\ Mx.$$

The categorical statement "All humans are mortal" would have to be symbolized

$$\forall x(Hx \supset Mx).$$

The above is literally read, "For all x, if x is human, then x is mortal." The expression

$$\forall x(Hx \& Mx)$$

is read, "For all x, x is both human and mortal." That is, it asserts that everything, not just humans, are mortal, which is obviously an incorrect interpretation of "All humans are mortal."

Instantiation

The interpretation of general statements and how they are instantiated or applied to individuals can be clarified if we assume a universe of discourse of only three individuals. Let us name these three individuals a, b, and c. If we which to attribute some predicate D to all individuals, we can assert

$$\forall x\ Dx$$

which, for our universe of discourse is materially equivalent[1] to

$$Da \ \& \ Db \ \& \ Dc.$$

If we wish to assert that every individual which has the predicate D also has the predicate E, then we would write

$$\forall x(Dx \supset Ex).$$

Applied to our limited universe of discourse, this statement expands to

$$(Da \supset Ea) \ \& \ (Db \supset Eb) \ \& \ (Dc \supset Ec).$$

In this limited universe of discourse, the existentially general statement

$$\exists x\ Dx$$

is expanded as

$$Da \vee Db \vee Dc.$$

That is, D is said to be predicatable of at least one individual. That could be a or b or c.

What we see above is that general statements, be they universal or existential, are interpreted as applying to the entire universe of discourse. Universally general statements are applied conjunctively while existentially general statements are applied disjunctively.

Exercise 14.2

Using the suggested notation, symbolize the statements expressed by the following sentences.

1. Everything is extended. (Ex: x is extended)
2. Everything is mental. (Mx: x is mental)
3. Everything is either extended or mental. (Ex: x is extended; Mx: x is mental)
4. No extended things are mental. (Ex: x is extended; Mx: x is mental)

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[1]The equivalence is material rather than logical. Logical equivalence would hold for any universe of discourse, not our limited universe of three individuals alone.

5. Things are extended if and only if they are not mental. (Ex: x is extended; Mx: x is mental)
6. Something is extended. (Ex: x is extended)
7. Nothing is mental. (Mx: x is mental)
8. All philosophers are logicians. (Px: x is a philosopher; Lx: x is a logician)
9. Some philosophers are logicians. (Px: x is a philosopher; Lx: x is a logician)
10. Some philosophers are not logicians. (Px: x is a philosopher; Lx: x is a logician)
11. No philosopher is both a mathematician and a logician.. (Px: x is a philosopher; Mx: x is a mathematician; Lx: x is a logician)
12. Some philosophers are mathematicians and some philosophers are logicians. (Px: x is a philosopher; Mx: x is a mathematician; Lx: x is a logician)
13. All philosophers are either logicians or mathematicians. (Px: x is a philosopher; Mx: x is a mathematician; Lx: x is a logician)
14. Only logicians are both mathematicians and philosophers. (Px: x is a philosopher; Mx: x is a mathematician; Lx: x is a logician)
15. Neither mathematicians nor logicians are philosophers. (Px: x is a philosopher; Mx: x is a mathematician; Lx: x is a logician)

Identity

A concept we use frequently in this and following chapters is that of identity, symbolized by "=". We will treat identity as an operator whose terms are individual constants or variables. Thus "$a = a$" and "$\forall x \forall y(x = y)$" are examples of well formed identity statements.

Self-Identity or Reflexivity

The law of identity, perhaps the most basic tautology, is that everything is identical to itself:

$$\forall x(x = x).$$

Any relation that a thing has to itself is reflexive.

Non-Identity

Non-identity may be symbolized in either of two ways: $\sim a=b$ or $a \neq b$. Since a statement of self-identity, $a=a$, is a tautology, any statement of non-self-identity, $a \neq a$, is a contradiction.

Symmetry

Identify is symetrical. Thus if $a=b$, then $b=a$. In formal notation,

$$\forall x \forall y(x=y \Leftrightarrow y=x).$$

Transitivity

Identity is also transitive. Thus if a=b and b=c, then a=c. In formal notation,

$$\forall x \forall y \forall x[(x=y \ \& \ y=z) \Rightarrow x=z]$$

Rules of Identity and Denied Identity

The rule of identity for reduction trees allows, given that one line of the tree is an identity statement, the substitution of one individual constant for the other in any other line. The application of the rule is illustrated in the following tree.

Aa	1. Aa	premise
Bb	2. Bb	premise
a=b	3. a=b	premise
Aa & Ba	√4. ~(Aa & Ba)	~ conclusion

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5. ~Aa      ~Ba   from 4, denied conjunction
     x    6.  Ba   from 2,3, identity
               x
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Note that line three, the identity statement, is not checked.

A statement of non-identity or denied identity may be symbolized in either of two ways. It may be expressed as ~a=b or as a≠b. Since any statement of self-identity, such as a=a, is a tautology, any denial of self-identity, such as a≠a, is a contradiction. Thus, whenever a line in a reduction tree is of the form, a≠a, the path may be closed. The following tree illustrates denied identity.

a = b	1. a = b	premise
c = d	2. c = d	premise
b ≠d	3. b ≠d	premise
a ≠c	√4.~a ≠ c	denied conclusion
	5. a = c	from 4, double denial
	6. b ≠ c	from 3, 2; identity
	7. b ≠ a	from 5,6; identity
	8. a ≠ a	from 7, 1; identity
	x	

Note that line 4 is truth functionally compound. The expression, ~a≠c , is logically equivalent to ~~a=c. Line 8, a≠a, is the contradiction which allows us to close the path.

Reduction Tree Rules for Quantifiers

We now need to add rules for general statements and the denial of general statements. Once we have done that, we will have completed the formal development of the predicate calculus.

Universal Instantiation (UI)

If a path contains a line that is universally general, then that line should be instantiated to every individual constant that occurs in the path. Do not check the line. Universally general statements are expanded conjunctively. They are true of every individual in the universe of discourse. Universal instantiation is illustrated in the following tree.

$\forall x(Ax \supset Bx)$
Aa
Ab
———
$Ba \,\&\, Bb$

1. $\forall x(Ax \supset Bx)$ premise
2. Aa premise
3. Ab premise
√ 4. $\sim(Ba \,\&\, Bb)$ ~conclusion
√ 5. $Aa \supset Ba$ from 1,UI
√ 6. $Ab \supset Bb$

7. $\sim Ba$ $\quad \sim Bb$ from 4, denied conjunction

8. $\sim Aa$ $\quad Ba$ from 5, UI $\quad \sim Ab$ $\quad Bb$ from 6, UI
x x x x

Since the individual constants, a and b, occur in lines 2 and 3, line 1 must be instantiated to both. Thus lines 5 and 6. Line 1 remains unchecked because, if additional constants are introduced, it will have to be instantiated to them as well.

Quantifier Negation (QN)

The rule for a negated quantifier is to change the quantity of the quantifier and to move the tilde from the left to the right of the quantifier. The equivalences for negated quantifiers are (using the Greek letter, Φ, as a predicate variable):

$$\sim\forall x\, \Phi x \;\Leftrightarrow\; \exists x \sim\Phi x$$
$$\sim\exists x\, \Phi x \;\Leftrightarrow\; \forall x \sim\Phi x$$

Check a line after applying the rule of quantifier negation. The rule for quantifier negation is used in the following example.

$\forall x(Ax \supset Bx)$
Ad
———
$\exists x\, Bx$

1. $\forall x(Ax \supset Bx)$ premise
2. Ad premise
√ 3. $\sim\exists x\, Bx$ ~conclusion
4. $\forall x \sim Bx$ from 3, QN
√ 5. $Ad \supset Bd$ from 1, UI
6. $\sim Bd$ from 4, UI

7. $\sim Ad$ $\quad Bd$ from 5, implication
x x

Existential Instantiation (EI)

An existentially general statement must be instantiated to an individual constant which has no previous occurance in the path. Existentially general statements are expanded disjunctively. They are true of at least one individual in the universe of discourse, but we do not know which one. After applying existential instantiation, check the line of the existentially general statement. Existential instantiation is illustrated in the tree below.

$\forall x(Ax \supset Dx)$
$\underline{\exists x\ Ax}$
$\exists x\ (Ax \ \&\ Dx)$

	1.	$\forall x(Ax \supset Dx)$	premise
√	2.	$\exists x\ Ax$	premise
√	3.	$\sim\exists x\ (Ax \ \&\ Dx)$	~conclusion
	4.	Aa	from 2, EI
	5.	$\forall x\sim(Ax \ \&\ Dx)$	from 3, QN
√	6.	$\sim(Aa \ \&\ Da)$	from 5, UI
√	7.	$Aa \supset Da$	from 1, UI

8. $\sim Aa$ $\sim Da$ from 6, denied conjunction
    x

9. $\sim Aa$ Da from 7, implication
          x   x

Note that EI is used before UI. Since existential instantiation must be to a constant which has no previous occurance in the path and universal instantiation must be to every constant which has an occurance in the path, to use UI first would be wasted effort.

Exercise 14.3

Construct reduction trees for the following valid arguments.

1. $\forall x(Sx \supset Tx)$
   $\underline{\exists x(Sx \ \&\ Rx)}$
   $\exists x(Rx \ \&\ Tx)$

2. $\forall x[(Sx \ \&\ Rx) \supset Tx]$
   $\forall x(Sx \supset Rx)$
   $\underline{\exists x\ Sx\qquad}$
   $\exists x\ Tx$

3. Sa
   Ra
   $\underline{\exists x(Sx \ \&\ Rx) \supset \exists x\ Tx\ }$
   $\exists x\ Tx$

4. $\exists x(Sx \vee Rx)$
   $\forall x(Sx \supset Tx)$
   $\underline{\forall x(Rx \supset Px)}$
   $\exists x(Tx \vee Px)$

5. Sa
   $\forall x(Sx \supset \sim Tx)$
   $\underline{\forall x(Tx \equiv Rx)\quad}$
   $Sa \ \&\ \sim Ra$

6. $\forall x[(Sx \ \&\ Tx) \supset x=a]$
   $\underline{\exists x[(Sx \ \&\ Tx) \ \&\ Rx]\ \ }$
   Ra

7. $\exists x\, Ex \supset \forall x(Ex \,\&\, Wx)$
$\underline{Ea\qquad\qquad\qquad}$
Wa

8. $\underline{Sa \,\&\, (Wa \vee Ta)\qquad}$
$\exists x[(Sx \,\&\, Wx) \vee (Sx \,\&\, Tx)]$

9. $\exists x(Sx \,\&\, Tx)$
$\underline{\exists x(Rx \,\&\, Px)}$
$\exists x(Sx \vee Rx)$

10. Sa
$\underline{Wb\qquad\qquad\qquad}$
$\exists x[a{=}b \supset (Sx \,\&\, Wx)]$

General Statements and Invalid Arguments

Invalid arguments can lead to more complicated trees than valid arguments. Consider the simple argument:

$$\begin{array}{l} \exists x\, Ax \\ \underline{\exists x\, Bx} \\ \exists x\, (Ax \,\&\, Bx) \end{array}$$

If we expand the argument for a universe of only one entity, "a," the argument is valid. But if the argument is expanded to a universe of two, it is invalid.

$$\begin{array}{lll} Aa & \qquad & Aa \vee Ab \\ \underline{Ba\qquad} & & \underline{Ba \vee Bb} \\ Aa \,\&\, Ba & & (Aa \,\&\, Ba) \vee (Ab \,\&\, Bb) \end{array}$$

Here is the reduction tree for the argument.

√ 1. $\exists x\, Ax$ premise
√ 2. $\exists x\, Bx$ premise
√ 3. $\sim\exists x\, (Ax \,\&\, Bx)$ ~ conclusion
4. $\forall x\ \sim(Ax \,\&\, Bx)$ 3, QN
5. Aa 1, EI
6. Bb 2, EI
√ 7. $\sim(Aa \,\&\, Ba)$ 4, UI
√ 8. $\sim(Ab \,\&\, Bb)$ 4, UI

9. $\sim Aa$ | $\sim Ba$ 7, denied conjunction
x

10. $\sim Ab$ | $\sim Bb$ 8, denied conjunction
x

Note that, had line 2 been instantiated to "a" (contrary to the rule for existential instantiation), both paths could have been closed at line 9. Thus the argument would have been valid for a universe of one, though not valid for a universe of two. If an argument is invalid for a universe of *n* entities, it is invalid for any universe of entities greater than *n*. The counter-example for the above argument is:

Aa	Ab	Ba	Bb
T	F	F	T

Exercise 14.4

Construct reduction trees for the following invalid arguments, and provide counter-examples from the trees.

1. $\forall x\ (Ax \supset Bx)$
   $\underline{\exists x\ Bx}$
   $\exists x\ Ax$

2. $\exists x\ (Ax \ \&\ Bx)$
   $\underline{\exists x\ (Ax \ \&\ {\sim}Bx)}$
   $\exists x\ (Bx \ \&\ {\sim}Bx)$

3. $\forall x\ (Ax \supset {\sim}Bx)$
   Aa
   $\underline{Bb}$
   $\exists x\ (Ax \ \&\ Bx)$

4. $\forall x\ [(Ax \ \&\ Bx) \supset Cx]$
   $\exists x\ Ax$
   $\underline{Ba}$
   $\exists x\ Cx$

5. $\forall x\ [{\sim}Ax \vee (Bx \ \&\ Cx)]$
   $\exists x\ Bx$
   $\underline{\exists x\ (Cx \ \&\ Dx)}$
   $\forall x\ {\sim}Ax$

6. Aa
   $\underline{Ba}$
   $\forall x\ (Ax \ \&\ Bx)$

7. $Aa \ \&\ \exists x\ (Bx \ \&\ x{=}a)$
   $\underline{Ab}$
   Bb

8. $\forall x\ [Ax \supset (Bx \vee Cx)]$
   $\exists x\ Ax$
   $\underline{\exists x\ Cx}$
   ${\sim}\exists x\ Bx$

9. $\forall x\ [(Ax \vee Bx) \supset Cx]$
   $\underline{\exists x\ Ax}$
   $\exists x\ (Ax \ \&\ Bx)$

10. $\forall x\ (Ax \supset Bx)$
    $\underline{\exists x\ (Ax \equiv Bx)}$
    $\exists x\ (Ax \ \&\ Bx)$

Exercise 14.5

Construct reduction trees for the following arguments, determining their validity or invalidity. If the argument is invalid, provide a counter-example.

1. $\exists x[Ax \,\&\, (Bx \vee Cx)]$
   $\underline{\forall x\,(Bx \supset x{=}a)}$
   $Ba \vee \exists x\,(Ax \,\&\, Cx)$

2. $\forall x\,(Ax \equiv Bx)$
   $\exists x\,(Bx \equiv Cx)$
   $\underline{{\sim}\exists x\,Cx}$
   ${\sim}\exists x\,Ax$

3. $Aa \,\&\, Ba$
   $\underline{\exists x\,(Ax \,\&\, Cx)}$
   $\exists x\,(Ax \,\&\, Bx)$

4. $\forall x\,(Ax \supset x{=}a)$
   $Ba \,\&\, a{=}b$
   $\underline{Ac}$
   Bc

5. $a \neq b$
   $b \neq c$
   ${\sim}Ac$
   $\underline{Ba}$
   ${\sim}Bc \,\&\, {\sim}\,Aa$

6. $\forall x\,(Ax \,\&\, Bx)$
   $\underline{Ca}$
   $(Aa \,\&\, Ba) \,\&\, Ca$

7. $\forall x\,{\sim}(Ax \,\&\, Bx)$
   $\underline{\exists x\,(Cx \equiv {\sim}Ax)}$
   $\exists x\,({\sim}Ax \,\&\, Cx)$

8. $\exists x\,(Ax \vee Bx)$
   $\forall x\,[(Ax \vee Bx) \supset Cx]$
   $\underline{\exists x\,Cx \supset Ca}$
   Ca

9. $\forall x\,(Ax \supset x{=}a)$
   $\forall x\,(Bx \supset x{=}b)$
   $\underline{Bc \,\&\, Ac}$
   $Aa \,\&\, Bb$

10. $\exists x\,(Ax \,\&\, Bx)$
    $\exists x\,(Bx \,\&\, Cx)$
    $\underline{\forall x\,(Bx \supset Cx)}$
    $\exists x\,(Ax \,\&\, Cx)$