STATISTICAL MECHANICS OF FAREY FRACTION SPIN CHAIN MODELS

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This thesis considers several statistical models defined on the Farey fractions. Two of these models, considered first, may be regarded as "spin chains", with long-range interactions, another arises in the study of multifractals associated with chaotic maps exhibiting intermittency. We prove that these models all have the same free energy. Their thermodynamic behavior is determined by the spectrum of the transfer operator (Ruelle-Perron-Frobenius operator), which is defined using the maps (presentation functions) generating the Farey "tree". The spectrum of this operator was completely determined by Prellberg. It follows that all these models have a second-order phase transition with a specific heat divergence of the form $C \sim [\epsilon \ln^2 \epsilon]^{-1}$. The spin chain models are also rigorously known to have a discontinuity in the magnetization at the phase transition.

The second part of this work extends our model by introducing an external field h. From rigorous and more heuristic arguments, we determine the phase diagram and

phase transition behavior of the extended model. Our results are fully consistent with scaling theory (for the case when a "marginal" field is present) despite the unusual nature of the transition for $h = 0$.

The third part of this thesis introduces a new family of partition functions with the same free energy. These models generalize one of the spin chain by introducing a new real parameter x. The structure of the Farey fractions then leads to a recurrence formula which has a direct connection to the operator studied by Prellberg. This connection provides a new and simple relation of the Contucci and Knauf "canonical" and "grand canonical" partition functions for any length of spin chain to the function obtained by the action of the operator on a constant function. In addition, we use the new partition functions to calculate certain expectation values and correlation functions.

DEDICATION

To my family and friends.

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TABLE OF CONTENTS

DEDICATION . . . . . **ii**

ACKNOWLEDGEMENTS . . . . . **iii**

LIST OF FIGURES . . . . . **vi**

Chapter

1 INTRODUCTION **1**

2 FREE ENERGY AND PHASE TRANSITION **7**

2.1 Partition functions defined on Farey fractions 7

2.2 Equivalence of the Farey tree and Knauf free energies 10

2.3 Transfer operator approach 13

2.4 Order of the phase transition and discussion 18

3 THERMODYNAMICS OF THE FAREY FRACTION SPIN CHAIN **21**

3.1 Definition of the model 21

3.2 Free energy with an external field 23

3.3 Renormalization group analysis 24

3.3.1 Mean field theory 24

3.3.2 Renormalization group analysis 25

3.3.3 Finite-size scaling 30

3.4 1-D KDP model with nonzero external field 31

4 TRANSFER OPERATOR, EXPECTATION VALUES, AND CORRELATION FUNCTIONS 37

4.1 Definition of the partition function 37

4.2 Spin orientation invariance and their consequences 39

4.3 Connection to the transfer operator 41

4.4 Expectation values and correlations 45

4.5 Infinitely long spin chain 48

5 CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK 53

REFERENCES 55

APPENDIX – BOUNDS FOR $\frac{Z_{N+1}}{Z_N}$ 57

BIOGRAPHY OF THE AUTHOR 59

LIST OF FIGURES

Figure 1 Free energy and magnetization vs. reduced temperature 4

Figure 2 Farey tree 8

Figure 3 Phase diagram 29

Figure 4 KDP 32

Figure 5 Phase diagram 34

Figure 6 Notation 35

CHAPTER 1

INTRODUCTION

In this work we consider several statistical models defined on the Farey fractions. The first is the Farey Fraction Spin Chain (FFSC), a one-dimensional statistical model first proposed by (Kleban and Özlük, 1999). This work has spawned a number of further studies, by both physicists and number theorists (Knauf, 1993; Contucci et al., 1999; Kallies et al., 2001; Peter, 2001). One can define the model as a periodic chain of sites with two possible spin states (A or B) at each site. The interactions are long-range, which allows a phase transition to exist in this one-dimensional system. The Farey spin chain is rigorously known to exhibit a single phase transition at temperature $\beta_c = 2$ (Kleban and Özlük, 1999). The phase transition itself is most unusual. The low temperature state is completely ordered (Kleban and Özlük, 1999; Contucci et al., 1999) . In the limit of a long chain, for $\beta > \beta_c$, the system is either all A or all B. Therefore the free energy is constant and the magnetization (defined via the difference in the number of spins in state A vs. those in state B) is completely saturated over this entire temperature range. Thus, even though the system has a phase transition at finite temperature, there are no thermal effects at all in the ordered state.

At temperatures above the phase transition (for $\beta < \beta_c$), fluctuations occur, and the free energy decreases with β. Here the system is paramagnetic. Since there is no symmetry-breaking field in the model, the magnetization vanishes. Thus the magnetization jumps from its saturated value in the low temperature phase, to zero in the high temperature phase (Contucci and Knauf, 1997). This might suggest a first-order phase transition, but the behavior with temperature is different. In Chapter 2, we prove that as a function of temperature, the transition is second-order,

and the same as that which occurs in the Knauf spin chain (see below) and the "Farey tree" multifractal model. The latter exhibits intermittency, and was studied by Feigenbaum, Procaccia, and Tél (Feigenbaum et al., 1989).

The Farey fraction spin chain is defined in an unusual way. It is given in terms of the energy of each possible configuration, rather than via a Hamiltonian. There is no known way to express the energy exactly in terms of the spin variables (Kleban and Özlük, 1999). Further, numerical results indicate that when one does, the Hamiltonian has all possible even interactions (and they are all ferromagnetic), so an explicit Hamiltonian representation, even if one could find it, would be exceedingly complicated.

In previous work (Kleban and Özlük, 1999), it was proven that the Farey spin chain free energy (per site, in the infinite chain limit) is the exactly same as the free energy of an earlier, related "number - theoretical" spin chain model due to Knauf (Knauf, 1993; Contucci and Knauf, 1997). In the present work, we extend this result in several ways.

In Chapter 2 we begin by defining the Farey fraction spin chain and Farey tree models. We than prove that the free energy for the Farey tree model is the same as the free energy of the Knauf model. This is established by use of bounds on the Knauf partition function. In Section 2.3, we examine the Farey model, which is specified by the maps (presentation functions (Feigenbaum et al., 1989)) that generate the Farey tree. The free energy in this case is given by the logarithm of the largest eigenvalue $\lambda(\beta)$ of the transfer operator. Some years ago, Knauf (Knauf, 1998) realized that the free energy of the Knauf model is also given by the logarithm of $\lambda(\beta)$, (without noting the connection to the Farey tree model, however). Combining his result with our analysis rigorously shows the equality of all four free energies-for the Farey fraction spin chain, Knauf model, Farey tree model and Farey model. In Section 2.4, by using the results of (Prellberg and Slawny, 1992), we show that the phase transition is

continuous (and of second order, i.e., the specific heat is divergent). It also follows that the phase transition in the Farey model occurs at the Hausdorff dimension of the Farey tree system, as expected. We conclude by briefly pointing out some connections with number theory and mentioning some implications of scaling theory for the spin chain models.

In Chapter 3 we regard the Farey fraction spin chain (FFSC) as a periodic chain of sites with two possible spin states (A or B) at each site. This model is rigorously known to exhibit a single phase transition at temperature $\beta_c = 2$ (Kleban and Özlük, 1999). The low temperature state is completely ordered (Kleban and Özlük, 1999; Contucci et al., 1999) . In the limit of a long chain, for $\beta > \beta_c$, the system is either all A or all B. Therefore the free energy f is constant and the magnetization m (defined via the difference in the number of spins in state A vs. those in state B) is completely saturated over this entire temperature range. Thus, even though the system has a phase transition at finite temperature, there are no thermal effects at all in the ordered state. The same thermodynamics occurs in the Knauf spin chain (KSC) (Knauf, 1993; Contucci and Knauf, 1997; Knauf, 1998; Guerra and Knauf, 1998), to which the FFSC is closely related.

At temperatures above the phase transition (for $\beta < \beta_c$), fluctuations occur, and f decreases with β. Here the system is paramagnetic, since (when the external field vanishes, see below) there is no symmetry-breaking field. Thus as the temperature increases m jumps from its saturated value in the ordered phase to zero in the high-temperature phase (Contucci and Knauf, 1997; Contucci et al., 1999) (see Fig. 1). (The KSC behaves similarly.)

One-dimensional models with long-range ferromagnetic interactions (Aizenman et al., 1988; Aizenman and Newman, 1986) are known to exhibit a discontinuity in m at β_c, but in these cases the jump in m is less than the saturation value.

The discontinuity in m might suggest a first-order phase transition, but in our model the behavior with temperature is different. However, Chapter 2 proves that as a function of temperature, f exhibits a second-order transition, and the same transition occurs in the KSC and the "Farey tree" multifractal model (Fiala et al., 2003). In beginning the research reported here, our motivation was to see whether the

Figure 1: Free energy and magnetization vs. reduced temperature

phase transition in the FFSC, which seems to mix first- and second-order behavior, is consistent with scaling theory. Indeed, as will be made clear, it is, in the "borderline" case when a marginal variable is present. In order to see this, we extend the definition of the FFSC to include a finite external field h. We then determine the phase diagram and free energy as a function of β and h, using both rigorous and renormalization group (RG) analysis.

In the following, Section 3.1 defines the model. Then, in Section 3.2 we prove the existence of the free energy f with an external field, and evaluate f for temperatures below the phase transition. In Section 3.3 we employ renormalization group arguments to find the free energy and phase diagram for temperatures above the phase transition. Section 3.4 considers a simple model that has very similar thermodynamics but is completely solvable. In Appendix we present some arguments needed to prove the existence of $f(\beta, h)$ in Section 3.2.

Since our results may be of interest to mathematicians who are unfamiliar with some of the physics employed herein, we pause to include a description of them from a more mathematical point of view. Section 3.1 defines the model and the quantities of interest. More specifically, the partition function Z_N is a two-parameter weighted sum over the (matrices defining the) Farey fractions, and the free energy f then follows

from the limiting procedure defined in (3.4). The main goal of our work is to find the analytic behavior of f as a function of the real parameters β, the inverse temperature (so $\beta > 0$ is implicit), and h, the external field. Regions of parameter space for which f is analytic are (thermodynamic) phases, and the lines of singularities that separate them are phase boundaries. In Section 3.2 we prove that $f(\beta, h)$ exists, and compute it exactly at low temperature (for $\beta > \beta_c$), which constitutes part of the ordered phase. Section 3.3 uses renormalization group methods to determine f at high temperatures (for β near β_c and $\beta < \beta_c$). Since this method is not rigorous, from a mathematical point of view the results should be regarded as conjectures. The main conclusions are the form of the free energy in the high-temperature phase (3.30, 3.31), the equation for the phase boundary (3.32, 3.33) and the change in magnetization $m = -\partial f/\partial h$ (3.35) and entropy $s = \beta^2\, \partial f/\partial \beta$ across the phase boundary. We also find that the ordered phase, with $f = \mp h$, extends to $\beta < \beta_c$ when h is sufficiently large (see Fig. 3). Section 3.3.3 gives predictions for the behavior of Z_N as $N \to \infty$ near the second-order point ($\beta = \beta_c$ and $h = 0$). This is related to some work in number theory, but unfortunately not yet directly. Section 3.4 examines an exactly solvable model with certain similarities to the FFSC.

In the Chapter 4, we extend the definition of the "number - theoretical" partition function studied by Knauf (see also (Feigenbaum et al., 1989; Artuso et al., 1989)) by introducing a parameter $x \in \mathbb{R}_0^+$. Both the "canonical" and "grand canonical" partition functions arise (Contucci and Knauf, 1997), for different values of x. More generally, this new parameter allows us to derive a recurrence relation on the length k of the spin chain. This recurrence formula is directly connected to the operator studied by Prellberg (cf. Chapter 2) and provides relations which can be used to calculate certain spin expectation values and correlations. The direct connection also provides more insight to the relation between the Prellberg operator and the operator studied by Contucci and Knauf. We explore how the spectrum of these two operators

are connected. At the end of this chapter we consider the finitely and infinitely long spin chains and calculate spin expectation values in the Knauf model. We also calculate the "edge" correlation length (left and right) for this model.

CHAPTER 2

FREE ENERGY AND PHASE TRANSITION

In this chapter we introduce several one-dimensional statistical models with long-range interaction defined on set of Farey fractions. We show that the free energy for all the models is the same.

In order to identify the order of the phase transition in all models, we discuss the connections of studied models to the transfer operator and its spectrum.

2.1 Partition functions defined on Farey fractions

We use the notation $r_k^{(n)} := \frac{n_k^{(n)}}{d_k^{(n)}}$ for the Farey fractions, where n is the order of the Farey fraction in level k. Level $k = 0$ consists of the two fractions $\left\{\frac{0}{1}, \frac{1}{1}\right\}$. Succeeding levels are generated by keeping all the fractions from level k in level $k+1$, and including new fractions. The new fractions at level $k+1$ are defined via $d_{k+1}^{(2n)} := d_k^{(n)} + d_k^{(n+1)}$ and $n_{k+1}^{(2n)} := n_k^{(n)} + n_k^{(n+1)}$, so that

$k = 0 \quad \left\{\frac{0}{1}, \frac{1}{1}\right\}$

$k = 1 \quad \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$

$k = 2 \quad \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$, etc.

Note that $n = 1, \ldots, 2^k + 1$. When the Farey fractions are defined using matrices (spin states) A and B, the level k corresponds to the number of matrices and hence the length of the spin chain (Kleban and Özlük, 1999).

It follows that the fractions in a given level are always in increasing order. The Farey fractions differ from the Farey "tree" (Feigenbaum et al., 1989), where only the new fractions are kept at each succeeding level (see Fig. 2.1).

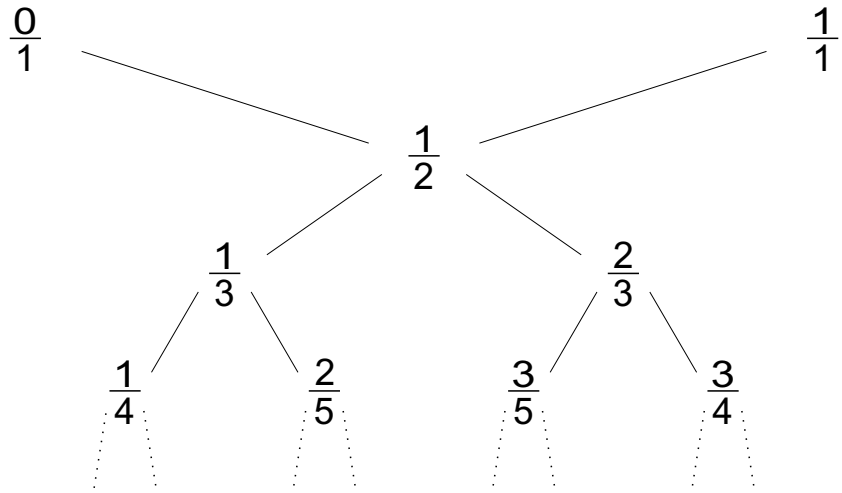

Figure 2: Farey tree

The partition function for the Farey fraction spin chain (we use just FC for superscript) may be written as (Kleban and Özlük, 1999)

$$Z_k^{FC}(\beta) := \sum_{n=1}^{2^k} \frac{1}{(d_k^{(n)} + n_k^{(n+1)})^\beta}, \quad \beta \in \mathbb{R}. \tag{2.1}$$

Note from (2.1) that there are 2^k states at level k with energies $E_k^{(n)} = \ln(d_k^{(n)} + n_k^{(n+1)})$. The Farey fractions (and hence the energies) can also be defined using the spin variables A and B mentioned above (Kleban and Özlük, 1999), but this is not needed here.

For present purposes, it is convenient to use the partition function for the Knauf model (Contucci and Knauf, 1997), which is rigorously known to have the same free energy as the Farey spin chain (Kleban and Özlük, 1999). The Knauf partition function may be defined via

$$Z_k^K(\beta) := \sum_{n=1}^{2^k} \frac{1}{(d_k^{(n)})^\beta}, \quad \beta \in \mathbb{R}, \tag{2.2}$$

so that a chain of length k has 2^k states of energy $E_k^{(n)} = \ln d_k^{(n)}$. The partition function can be written as sum of even and odd terms

$$Z_k^K(\beta) = Z_{k,e}^K(\beta) + Z_{k,o}^K(\beta), \tag{2.3}$$

where

$$Z^K_{k,e}(\beta) := \sum_{n=1}^{2^{k-1}} \frac{1}{(d_k^{(2n)})^\beta}, \quad Z^K_{k,o}(\beta) := \sum_{n=1}^{2^{k-1}} \frac{1}{(d_k^{(2n-1)})^\beta}.$$

From the definition of the Farey fractions immediately follows

$$d_k^{(2n)} = d_k^{(2n-1)} + d_k^{(2n+1)} \tag{2.4}$$

and

$$d_k^{(2n-1)} = d_{k-1}^{(n)}. \tag{2.5}$$

From (4.26) we have

$$d_k^{(2n)} > d_k^{(2n-1)}, \qquad d_k^{(2n)} > d_k^{(2n+1)},$$

while from (2.5) we obtain $Z^K_{k,o}(\beta) = Z^K_{k-1}(\beta)$ so that

$$Z^K_{k,e}(\beta) = Z^K_k(\beta) - Z^K_{k-1}(\beta). \tag{2.6}$$

The Farey tree model of Feigenbaum, Procaccia and Tél (Feigenbaum et al., 1989) uses the "Farey tree" rather than the Farey fractions, which means retaining only the 2^{k-1} even fractions at level $k > 1$ so we obtain the set

$$\{r_k^{(2n)} | \, n = 1, \ldots, 2^{k-1}, k > 1\}.$$

The Farey tree partition function is defined by

$$Z^F_k(\beta) := \sum_{n=1}^{2^{k-2}} \left(r_k^{(4n)} - r_k^{(4n-2)} \right)^\beta. \tag{2.7}$$

The positive quantities $\left(r_k^{(4n)} - r_k^{(4n-2)} \right)$ are the radii of the "balls" in this model. Note that we can also express this partition function using Farey tree denominators only. One finds

$$Z^F_k(\beta) = \sum_{n=1}^{2^{k-2}} \left(\frac{3}{d_k^{(4n)} d_k^{(4n-2)}} \right)^\beta.$$

2.2 Equivalence of the Farey tree and Knauf free energies

In this section, we show the equivalence of the free energies of the Knauf and Farey tree models. We begin by finding bounds for the Farey tree partition function $Z_k^F(\beta)$ in terms of the Knauf partition function. We are interested in the case $\beta > 0$, where there is a phase transition, but it will be easy to see that the free energies are equal for all $\beta \in \mathbb{R}$.

The Farey fractions satisfy $r_k^{(n)} - r_k^{(n-1)} = 1/(d_k^{(n)} d_k^{(n-1)})$. This may be shown for instance using the matrix chain representation in (Kleban and Özlük, 1999). Thus

$$\begin{aligned} r_k^{(4n)} - r_k^{(4n-2)} &= r_k^{(4n)} - r_k^{(4n-1)} + r_k^{(4n-1)} - r_k^{(4n-2)} \\ &= \frac{1}{d_k^{(4n)} d_k^{(4n-1)}} + \frac{1}{d_k^{(4n-1)} d_k^{(4n-2)}} \\ &> \frac{1}{\left(d_k^{(4n)}\right)^2}, \end{aligned} \tag{2.8}$$

and similarly $r_k^{(4n)} - r_k^{(4n-2)} > 1/\left(d_k^{(4n-2)}\right)^2$. From (2.8) we also find

$$r_k^{(4n)} - r_k^{(4n-2)} < \frac{2}{\left(d_k^{(4n-1)}\right)^2}. \tag{2.9}$$

Using (2.7) and (2.8), for $\beta > 0$, gives

$$Z_k^F(\beta) > \sum_{n=1}^{2^{k-2}} \frac{1}{\left(d_k^{(4n)}\right)^{2\beta}}, \tag{2.10}$$

and also $Z_k^F(\beta) > \sum_{n=1}^{2^{k-2}} 1/\left(d_k^{(4n-2)}\right)^{2\beta}$. Adding these two inequalities we find a lower bound for the Feigenbaum partition function

$$Z_k^F(\beta) > \frac{1}{2} \sum_{n=1}^{2^{k-1}} \frac{1}{\left(d_k^{(2n)}\right)^{2\beta}} = \frac{1}{2} Z_{k,e}^K(2\beta). \tag{2.11}$$

Using the inequality (2.9) and the relation (2.5) gives the upper bound

$$Z_k^F(\beta) < 2^\beta \sum_{n=1}^{2^{k-2}} \frac{1}{\left(d_k^{(4n-1)}\right)^{2\beta}} = 2^\beta \sum_{n=1}^{2^{k-2}} \frac{1}{\left(d_{k-1}^{(2n)}\right)^{2\beta}} = 2^\beta Z_{k-1,e}^K(2\beta). \tag{2.12}$$

Thus the Farey tree partition function at β is bounded both above and below by the even part of the Knauf partition function at 2β.

$$\frac{1}{2}Z^K_{k,e}(2\beta) < Z^F_k(\beta) < 2^\beta Z^K_{k-1,e}(2\beta), \quad \beta > 0 \tag{2.13}$$

Similarly, we can find, that

$$2^\beta Z^K_{k-1,e}(2\beta) < Z^F_k(\beta) < \frac{1}{2}Z^K_{k,e}(2\beta), \quad \beta < 0. \tag{2.14}$$

Finally, for $\beta = 0$ it is obvious that

$$Z^F_k(\beta) = \frac{1}{4}Z^K_k(2\beta).$$

The free energy per site is defined by

$$f(\beta) := \frac{-1}{\beta}\lim_{k\to\infty}\frac{\ln Z_k(\beta)}{k}. \tag{2.15}$$

(Recall that the level k corresponds to the length of the spin chain.) We now use (2.13) to prove that

$$f_F(\beta) = f_K(2\beta).$$

where f_F refers to the free energy obtained from Z^F_k.

For $\beta > 1$ one has (Knauf, 1993)

$$Z^K_k(2\beta) \stackrel{k\to\infty}{\longrightarrow} \frac{\zeta(2\beta-1)}{\zeta(2\beta)},$$

which implies that $f_K(2\beta) = 0$. Also, by (2.6),

$$Z^K_{k,e}(2\beta) \stackrel{k\to\infty}{\longrightarrow} 0,$$

and using (2.13) gives

$$Z^F_k(\beta) \stackrel{k\to\infty}{\longrightarrow} 0.$$

Since $Z^F_k(\beta) > 0$,

$$\frac{-\ln Z^F_k(\beta)}{k} \geq 0 \Rightarrow f_F(\beta) \geq 0.$$

Note that for $\beta = 1$ one has $Z_k^F(1) \leq 1$, since this partition function reduces to a simple sum of Farey tree fraction separations (ball lengths), which cannot exceed the length of the interval $[0, 1]$. Therefore the inequality still holds (and in fact, as shown below, $f_F(1) = 0$).

Now clearly

$$Z_{k,e}^K(2\beta) > \frac{1}{(k+1)^{2\beta}}$$

so by (2.13) we find

$$Z_k^F(\beta) > \frac{1}{2}\frac{1}{(k+1)^{2\beta}},$$

and

$$0 \leq \frac{-\ln Z_k^F(\beta)}{k} < \frac{2\beta \ln(k+1)}{k} + \frac{\ln 2}{k}.$$

Thus we have

$$f_F(\beta) = f_K(2\beta) = 0 \text{ for } \beta \geq 1. \tag{2.16}$$

The validity of $f_K(2) = 0$ is clear from the treatment in (Knauf, 1998) and the remark at the end of this section.

For $\beta < 1$ we can write

$$Z_{k,e}^K = Z_k^K - Z_{k-1}^K = Z_k^K \left(1 - \frac{Z_{k-1}^K}{Z_k^K}\right),$$

so

$$-\frac{\ln Z_{k,e}^K}{k} = -\frac{\ln Z_k^K}{k} - \frac{\ln\left(1 - \frac{Z_{k-1}^K}{Z_k^K}\right)}{k}. \tag{2.17}$$

It is shown in (Contucci and Knauf, 1997) (by arguments using the transfer operator, see below) that for $0 < \beta < 1$ the free energies obtained from Z_k^K and $Z_{k,e}^K$ are the same, thus for $k \to \infty$

$$\frac{\ln\left(1 - \frac{Z_{k-1}^K}{Z_k^K}\right)}{k} \to 0. \tag{2.18}$$

(This also can be shown directly by considering the equation

$$Z_k^K(2\beta) = 1 + \sum_{j=1}^{k} Z_{j,e^K}(2\beta),$$

which follows from (2.6). For $0 < \beta < 1$ the series is bounded by a geometric series because of the inequality $Z^K_{k,e} > 2^{1-\beta} Z^K_{k-1,e}$.) For $\beta \leq 0$ it is easy to check that $Z^K_{k-1,e}(2\beta)/Z^K_{k,e}(2\beta) \leq 1/2$. Thus (2.18) holds for all $\beta < 1$.

Using (2.13) (and, for $\beta \leq 0$, (2.14) and the line below) then establishes

$$f_F(\beta) = f_K(2\beta) \text{ for } \beta < 1. \tag{2.19}$$

Note that, as mentioned, the Knauf partition function $Z^K_k(2\beta)$ is finite as $k \to \infty$ for $\beta > 1$ (Knauf, 1993). Using (2.6) and (2.13) one sees immediately that the Farey tree partition function $Z^F_k(\beta)$ vanishes in this limit for $\beta > 1$. At $\beta = 1$, it follows immediately from the definition (2.7) and simple properties of the Farey fractions that $0 < Z^F_k(1) < 1$. For $\beta < 1$, since $f_K(2\beta) < 0$ (Contucci and Knauf, 1997) and using (2.19) and (4.31) it follows that $Z^F_k(\beta)$ is infinite. This establishes rigorously that the Hausdorff dimension of the set formed by the "balls" is $\beta_H = 1$, as expected.

Finally, consider (2.13) and the fact, mentioned above, that $Z^F_k(1) < 1$. It follows that

$$Z^K_{k,e}(2) = \sum_{n=1}^{2^{k-1}} \frac{1}{(d_k^{(2n)})^2} < 2,$$

so that this sum over the "new" Farey denominators is bounded by 2 at all levels. Since the "new" denominators at level $k-1$ become "old" denominators at level k, one also sees that $Z^K_k(2) \leq 2k+1$.

2.3 Transfer operator approach

In this section we consider the transfer operator (Ruelle-Perron-Frobenius operator) of the Farey map. The previous section shows rigorously that the free energies of the Knauf and Farey fraction spin chain and Farey tree model are the same. Here we prove that they (as well as the free energy of the Farey tree model in a certain approximation specified below) are simply given by the largest eigenvalue of this operator. The next section considers the asymptotic behavior of this eigenvalue near the phase transition,

known from the work of Prellberg (Prellberg, 2003), which specifies the order of the phase transition.

The Ruelle-Perron-Frobenius operator $\mathcal{K}$ associated with a map f (piecewise monotonic transformation of closed interval I) is given by

$$\mathcal{K}_\beta \varphi(x) = \sum_{f(y)=x} |f'(y)|^{-\beta} \varphi(y), \quad \beta \in \mathbb{R}, \tag{2.20}$$

where the sum is over each strictly monotonic and continuous piece of f satisfying the summation condition. See (Prellberg and Slawny, 1992; Prellberg, 1991) for a more complete discussion.

The Farey map is defined by (Feigenbaum et al., 1989; Prellberg and Slawny, 1992)

$$f(x) = \begin{cases} f_0(x) = x/(1-x), & 0 \le x \le 1/2, \\ f_1(x) = (1-x)/x, & 1/2 < x \le 1. \end{cases} \tag{2.21}$$

The operator then consists of two corresponding terms $\mathcal{K}_0$ and $\mathcal{K}_1$ which can be identified as "intermittent" and "chaotic" parts, respectively (Prellberg, 2003). We may write $\mathcal{K}_\beta = \mathcal{K}_0 + \mathcal{K}_1$ where $\mathcal{K}_i \varphi(x) = |F_i'(x)|^\beta \varphi(F_i(x))$ and the "presentation function" (Feigenbaum et al., 1989) F_i is the inverse map of f_i (see (2.30) below). Thus

$$\mathcal{K}_\beta \varphi(x) = (1+x)^{-2\beta} \left[\varphi\left(\frac{x}{1+x}\right) + \varphi\left(\frac{1}{1+x}\right) \right], \quad \beta \in \mathbb{R}. \tag{2.22}$$

Following the thermodynamic formalism approach (Ruelle, 1978) it was shown in (Prellberg and Slawny, 1992; Prellberg, 1991) that the largest eigenvalue of $\mathcal{K}_\beta$ in (2.22) (defined on the space of functions with bounded variation) is related to a free energy via $f(\beta) = -\beta^{-1} \ln \lambda(\beta)$ for $\beta \in \mathbb{R}$. We call this the free energy of the Farey model.

In this section we consider $\mathcal{K}_\beta$ acting on L^2 and show that the free energy obtained from its largest eigenvalue is the same as the free energy of the Knauf and Farey tree model (in its original version or using the approximation below) for $0 < \beta < 1$. In

the next section, we prove that the free energy of the Farey model in this β range is also the same. For $\beta > 1$, the free energy of any of these models is already known to be zero (see 2.2 or (Prellberg, 2003)).

The Knauf spin chain at level $k-1$ may be described by a vector $Y_{k-1}(2\beta) \in l^2(\mathbb{N}_0)$, the first component of which is the "even" Knauf partition function $Z^K_{k,e}(2\beta)$. The "transfer operator" of the Knauf spin chain then maps $Y_{k-1}(2\beta)$ to the next level:

$$Y_k(2\beta) = \tilde{\mathcal{C}}(2\beta) Y_{k-1}(2\beta), \tag{2.23}$$

where $\tilde{\mathcal{C}}(2\beta) : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ and (Contucci and Knauf, 1997)

$$\tilde{C}(2\beta)_{i,j} = (-1)^j \, 2^{-2\beta-i-j} \left[\begin{pmatrix} -2\beta - i \\ j \end{pmatrix} + \sum_{s=0}^{i} 2^s \begin{pmatrix} i \\ s \end{pmatrix} \begin{pmatrix} -2\beta - i \\ j - s \end{pmatrix} \right], \tag{2.24}$$

$(i, j \in \mathbb{N}_0)$, with the generalized binomial coefficients $\begin{pmatrix} a \\ b \end{pmatrix} = (\Pi_{i=1}^{b-1}(a-i))/b!$, $a \in \mathbb{R}$, $b \in \mathbb{N}_0$, and $\begin{pmatrix} a \\ b \end{pmatrix} = 0$ if $b < 0$. Knauf (Knauf, 1998) has further shown that for $0 < \beta < 1$, $\tilde{\mathcal{C}}(2\beta)$ has the same largest eigenvalue $\lambda(\beta)$ as $\mathcal{K}_\beta : L^2((0,1)) \to L^2((0,1))$. The argument involves expanding (2.22) about $x = 1$ with $\varphi(x) = \sum_{m=0}^{\infty} a_m (1-x)^m$. Doing this, one finds that the action of $\mathcal{K}_\beta$ on the quantities a_m (note that $a_m = (-1)^m \varphi^{(m)}(1)/m!$) is given by $\tilde{\mathcal{C}}^T(2\beta)$, where T denotes transpose.

In addition, $\tilde{\mathcal{C}}^T(2\beta)$ is independent of k, so the components of the vector $X_k(2\beta)$ (defined using (2.23) with $\tilde{\mathcal{C}}^T(2\beta)$ replacing $\tilde{\mathcal{C}}(2\beta)$) are proportional to the Taylor series coefficients of an associated function $\phi_k^{(\beta)}(x)$. This function therefore satisfies

$$\phi_k^{(\beta)}(x) = (1+x)^{-2\beta} \left[\phi_{k-1}^{(\beta)}\left(\frac{x}{1+x}\right) + \phi_{k-1}^{(\beta)}\left(\frac{1}{1+x}\right) \right]. \tag{2.25}$$

It is shown in (Contucci and Knauf, 1997) that $\tilde{\mathcal{C}}(2\beta)$ (and hence $\tilde{\mathcal{C}}^T(2\beta)$) is an operator of Perron-Frobenius type for $0 < \beta < 1$. Thus $\lambda(\beta)$ is a simple eigenvalue (the same for $\tilde{\mathcal{C}}$ or $\tilde{\mathcal{C}}^T$). The corresponding eigenvector is strictly positive and unique,

and may be obtained (for $\tilde{\mathcal{C}}^T$) via $V(2\beta) = \lim_{k\to\infty} X_k(2\beta)/||X_k(2\beta)||$. In addition, it follows that for $0 < \beta < 1$ the eigenvalue $\lambda(\beta) > 1$ is an analytic function of β, and its positive normalized eigenvector $V(2\beta)$ is analytic in β. Hence

$$\phi_k^{(\beta)} \sim \lambda(\beta)^k \phi^{(\beta)}, \tag{2.26}$$

where $\phi^{(\beta)}(x)$ is the normalized eigenvector of $\mathcal{K}_\beta : L^2((0,1)) \to L^2((0,1))$ corresponding to $V(2\beta)$. Substituting this result in (2.25) we get, for $0 < \beta < 1$,

$$\lambda(\beta)\phi^{(\beta)}(x) = (1+x)^{-2\beta}\left[\phi^{(\beta)}\left(\frac{x}{1+x}\right) + \phi^{(\beta)}\left(\frac{1}{1+x}\right)\right], \tag{2.27}$$

which is equivalent to (2.22) when $\lambda(\beta)$ is the maximal eigenvalue. Then

$$\lim_{k\to\infty} \frac{Z_{k,e}^K(2\beta)}{Z_{k-1,e}^K(2\beta)} = \lambda(\beta) \tag{2.28}$$

together with (4.31), (2.17) and (2.18) give us the Knauf free energy as expected

$$f_K(2\beta) = -\frac{1}{\beta}\ln\lambda(\beta), \quad 0 < \beta < 1. \tag{2.29}$$

Note that for $\beta \geq 1$, $f_K(2\beta) = 0$ (see 2.2) and also that $f(\beta) = 0$ for $\beta \geq 1$ follows from the spectrum of the operator $\mathcal{K}_\beta$ ((Prellberg, 2003), see also the next section). Thus the free energy of the Farey spin chain, Farey tree and Knauf models are given by the largest eigenvalue of the Ruelle-Perron-Frobenius operator for $\beta > 0$.

To further examine these connections we follow the treatment in (Feigenbaum et al., 1989). We focus on (2.27) and make use of presentation functions. The Farey tree can be generated by two presentation functions

$$F_0 = \frac{x}{1+x}, \qquad F_1 = 1 - F_0 = \frac{1}{1+x}. \tag{2.30}$$

Every fraction at each level $k > 1$ of the Farey tree can be reached by composition of k functions F_ϵ ($\epsilon \in \{0,1\}$) evaluated at $x^* = \frac{1}{2}$. For example, at level $k = 3$, $F_0 \circ F_1(\frac{1}{2}) = \frac{2}{5} = r_3^{(4)}$. So the diameter of every "ball" in the Farey tree model (see (2.7)) can be written as

$$r_k^{(4n)} - r_k^{(4n-2)} = |F_{\epsilon_1} \circ F_{\epsilon_2} \circ \ldots \circ F_{\epsilon_{k-1}}(F_0(x^*)) - F_{\epsilon_1} \circ F_{\epsilon_2} \circ \ldots \circ F_{\epsilon_{k-1}}(F_1(x^*))|. \tag{2.31}$$

Note that the sequence of presentation functions in the two Farey fractions in (2.31) is identical except for the F_{ϵ_k}, i.e. only the presentation functions applied first to x^* differ. As $k \to \infty$, the diameter of the balls converges to zero (this follows easily from (2.8)). Therefore it is reasonable to suppose that for k sufficiently large each diameter can be approximated by the derivative of the composed function with respect to x^*. Then, using the chain rule, (2.31) behaves asymptotically as

$$r_k^{(4n)} - r_k^{(4n-2)} \sim |F'_{\epsilon_1}(F_{\epsilon_2} \circ F_{\epsilon_3} \circ \ldots)F'_{\epsilon_2}(F_{\epsilon_3} \circ F_{\epsilon_4} \circ \ldots) \ldots|. \tag{2.32}$$

Thus we can write for the partition function

$$Z_k^F \sim \ldots \sum_{\epsilon_k} |F'_{\epsilon_k}(F_{\epsilon_{k+1}} \circ F_{\epsilon_{k+2}} \circ \ldots)|^\beta \sum_{\epsilon_{k-1}} |F'_{\epsilon_{k-1}}(F_{\epsilon_k} \circ F_{\epsilon_{k+1}} \circ \ldots)|^\beta \ldots \tag{2.33}$$

Notice that the sum over ϵ_k and all lower indexed sums to its right depend only upon $(F_{\epsilon_{k+1}} \circ F_{\epsilon_{k+2}} \circ \ldots)$. This motivates the definition

$$\psi_k^{(\beta)}(x) := \sum_{\epsilon_k} |F'_{\epsilon_k}(x)|^\beta \sum_{\epsilon_{k-1}} |F'_{\epsilon_{k-1}}(F_{\epsilon_k}(x))|^\beta \ldots, \tag{2.34}$$

where $(F_{\epsilon_k} \circ F_{\epsilon_{k+1}} \circ \ldots)$ is denoted by x. One then finds

$$\psi_k^{(\beta)}(x) = \sum_{\epsilon} |F'_\epsilon(x)|^\beta \psi_{k-1}^{(\beta)}(F_\epsilon(x)). \tag{2.35}$$

Note that since each presentation function F_ϵ is a ratio of polynomials, one can extend the definition of $\psi_k^{(\beta)}(x)$ to the whole interval $[0, 1]$. Substituting for F and F' we obtain (2.25) (with ψ_k replacing ϕ_k). Therefore choosing $\psi_0^{(\beta)}(x) > 0$ we find $\psi_k^{(\beta)} \to \psi^{(\beta)}$ as $k \to \infty$, with the function $\psi^{(\beta)}$ proportional to $\phi^{(\beta)}$ (the eigenfunction with the maximum eigenvalue $\lambda(\beta)$). This establishes that the approximation (2.32) is exact in the limit $k \to \infty$, as expected.

Finally, it is interesting to note some connections with number theory. Specifically, for $\lambda = 1$, (2.27) is known as the Lewis equation and has been studied (for complex β) because of its connection to the Selberg ζ-function and period polynomials (cusp forms of the modular group) (Lewis and Zagier, 2001). An operator related to $\mathcal{K}_\beta$ (2.22) also appears in this context and is called the Mayer operator (Mayer, 1991).

2.4 Order of the phase transition and discussion

In the preceding, we have shown that the Farey spin chain (Kleban and Özlük, 1999), the Knauf spin chain (Contucci and Knauf, 1997) and (either version of) the Farey tree model (Feigenbaum et al., 1989) all have the same free energy. Further, for $0 < \beta < 1$ their free energy is given by the largest eigenvalue of the Farey model transfer operator acting on L^2 (2.22). Here we show that the transfer operator acting on the space of functions of bounded variation has the same leading eigenvalue in this β range, which allows us to make use of the results of Prellberg. The corresponding equality of free energies for $\beta > 1$ (where the free energy vanishes) follows from known results, as remarked in the previous section.

Prellberg has examined the spectrum of the operator $\mathcal{K}_\beta$ acting on the space of functions with bounded variation (Prellberg and Slawny, 1992) (details are in (Prellberg, 1991)). In order to make use of his results, we must show that the largest eigenvalue in this space is the same as that in $L^2((0,1))$. To prove this we examine the corresponding eigenvectors. Expanding $\varphi(x)$, the eigenvector in the L^2 space, about $x = 1$ as above, one has $\varphi(x) = \sum_{m=0}^{\infty} a_m (1-x)^m$. Thus $\varphi(1)$ is finite, since the coefficients a_m in this expansion are proportional to the components of the eigenvector of $\tilde{C}^T(2\beta)$ of largest eigenvalue (see Section 2.3). Furthermore, the a_m are all positive, since the eigenvector of $\tilde{C}^T$ is positive. Therefore $\varphi(x)$ is a (strictly) decreasing function on $[0,1]$. Finally, setting $x = 0$ in (2.27) shows that $\varphi(0)$ is finite whenever $\lambda \neq 1$. Therefore, $\varphi(x)$ is of bounded variation for $0 < \beta < 1$, and since both eigenvectors are unique (up to multiplicative constants) their eigenvalues must coincide in this range of β values.

The result of Prellberg of interest here is

$$\beta f(\beta) = c\frac{1-\beta}{\ln(1-\beta)}\left[1 + o(1)\right], \quad 0 < \beta < 1,$$

where $c > 0$, and $\beta f(\beta) = 0$ for $\beta \geq 1$. This form for the free energy is equivalent to that given in (Feigenbaum et al., 1989), as may be seen by use of the Lambert W-function.

The non-analyticity at $\beta = 1$ results in a phase transition of second order, since the second derivative of $f(\beta)$ diverges as $\left[(1-\beta)(\ln(1-\beta))^2\right]^{-1}$ as $\beta \to 1^-$. This result agrees with (Contucci and Knauf, 1997), where it is proven rigorously that the phase transition is at most second order. Note that the largest eigenvalue is discrete for $\beta < 1$. For $\beta > 1$, the discrete spectrum disappears and the largest eigenvalue becomes $\lambda = 1$, which is the upper boundary of the continuous spectrum for all β.

Our result for the free energy also has some implications for the number of states of the spin chain models. The Knauf model partition function may be expressed as a Dirichlet series (Knauf, 1993)

$$Z_k^K(\beta) = \sum_{n=1}^{\infty} \phi_k(n) n^{-\beta}, \tag{2.36}$$

where $\phi_k(n)$ is non-zero when n is a Farey denominator at level k. This function converges from below to the Euler totient function $\phi(n)$ as $k \to \infty$. Since the energy of an allowed state is $E = \ln n$, $\phi_k(n)$ gives the number of states of energy E at level k. The functions ϕ_k and ϕ are very irregular. Our result for the free energy then shows how the Dirichlet series in (2.36) diverges as $k \to \infty$ for small (but positive!) $(2-\beta)$. (Recall that the phase transition in the spin chains appears at $\beta_c = 2$, since a factor of 2 appears in comparing with the Farey tree model, see (2.29).) For the Farey spin chain, an equation with the same form as (2.36) may also be written, with the same leading divergent behavior. Here the limit of the function corresponding to $\phi_k(n)$ is not known, though some related information is available (Peter, 2001).

One can also consider the implications of scaling theory for the two spin chain models. It is known that the magnetization (defined via the difference in the number of spins in state A vs. those in state B) is one for temperatures below the transition

and zero above it (Contucci et al., 1999; Contucci and Knauf, 1997). Thus the magnetization jumps from its fully saturated value to zero at the transition. This would lead one to suspect a first-order transition, but as we have seen, the behavior with temperature is second-order. However, both these results seem to be consistent with scaling theory, with renormalization group eigenvalues $y_T = d$ and $y_h = d$, where d is the dimensionality, and using $(2-\beta)/\ln(2-\beta)$ as the temperature scaling variable. We examine this in the next chapter.

CHAPTER 3

THERMODYNAMICS OF THE FAREY FRACTION SPIN CHAIN

In this chapter we extend the spin chain models by introducing a magnetic field. Using rigorous results and renormalization group arguments we predict the phase diagram of these models and show that it is consistent with the scaling theory.

3.1 Definition of the model

The Farey fraction spin chain (FFSC) consists of a periodic chain of N sites (see Section 2.1 and note that number of sites N corresponds to the level of the Farey fractions k) with two possible spin states (A or B) at each site. The interactions are long-range, which allows a phase transition to exist in this one-dimensional system. Let the matrices

$$M_N := \prod_{i=1}^{N} A^{1-\sigma_i} B^{\sigma_i}, \qquad \sigma_i \in \{0,1\}, \tag{3.1}$$

where $A := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and the dependence of M_N on $\{\sigma_i\}$ has been suppressed. The energy of a particular configuration with N spins in an external field h is then given as

$$E_N := \ln(T_N) + h\left(2\sum_{i=1}^{N}\sigma_i - N\right) \quad \text{with} \quad T_N := \mathrm{Tr}(M_N). \tag{3.2}$$

Thus our partition function is

$$Z_N(\beta, h) = \sum_{\{\sigma_i\}} \mathrm{Tr}(M_N)^{-\beta} e^{-\beta h\left(2\sum_{i=1}^{N}\sigma_i - N\right)}. \tag{3.3}$$

Note that $Z_N(\beta, 0) = 2Z_N^{FC}(\beta)$, where the factor 2 follows from definition of eq. 2.1 and Lemma .1 (see Appendix, so this definition extends the Farey fraction spin chain

model to non-vanishing external field h. Given the nature of the low-temperature $h = 0$ system, it is natural to introduce h in this way.

The free energy is defined as

$$f(\beta, h) := \frac{-1}{\beta} \lim_{N\to\infty} \frac{\ln Z_N(\beta, h)}{N}. \tag{3.4}$$

The existence of the free energy $f(\beta, h)$ follows from simple bounds using $f(\beta, 0)$ (see section 3.2 below).

The definition of the FFSC is somewhat unusual. The partition function is given in terms of the energy of each possible configuration, rather than via a Hamiltonian. In fact, there is no known way to express the energy exactly in terms of the spin variables (Kleban and Özlük, 1999). Further, numerical results indicate that when one does, the Hamiltonian has all possible even interactions (and they are all ferromagnetic), so an explicit Hamiltonian representation, even if one could find it, would be exceedingly complicated.

Note that for $h = 0$ there are two ground states with energy $E = \ln 2$. The other $2^N - 2$ states have energy $\ln N \leq E \leq Nc$, where c is a constant. Therefore the difference between the lowest excited state energy and the ground state energy diverges as $N \to \infty$.

The phase transition in this system (Kleban and Özlük, 1999) occurs in the following way. Divide the partition function into two terms, one due to the two ground states, and the other (call it Z'), due to the remaining $2^N - 2$ states. The system remains in the ground states, and $Z' \to 0$ as $N \to \infty$, until the temperature is high enough that Z' diverges with N. In section 3.4 we examine a simple model that also exhibits this feature, but is completely solvable.

Our results also apply to the KSC, which has the same thermodynamics as the FFSC model at $h = 0$ (see Chapter 2 and (Fiala et al., 2003)). An external field may be included in the KSC in exactly the same way as described above for the FFSC. The "Farey tree" model of Feigenbaum et. al. (Feigenbaum et al., 1989) also has the

same free energy, but it is not clear how to incorporate a field h. Our finite-size results (see section 3.3.3) do apply when $h = 0$, however.

3.2 Free energy with an external field

In this section we show rigorously that $f(\beta, h)$ exists and that

$$f(\beta, h) = -|h|, \tag{3.5}$$

for $\beta > \beta_c$.

For $h > 0$ it is easy to see (from (3.3)) that

$$2^{-\beta} e^{\beta h N} < Z_N(\beta, h) < Z_N(\beta, 0) e^{\beta h N}. \tag{3.6}$$

Using the definition of the free energy then gives

$$-h \geq f(\beta, h) \geq f(\beta, 0) - h, \tag{3.7}$$

where $f(\beta, h)$ is understood to be defined via (3.4). Now $f(\beta, 0)$ is rigorously known to exist (Kleban and Özlük, 1999). In addition, we know that $f(\beta, 0) = 0$ for $\beta \geq \beta_c$ (Kleban and Özlük, 1999), which implies (3.5) for $h > 0$ ($h < 0$ follows similarly).

To see that $f(\beta, h)$ exists for the range $0 \leq \beta < \beta_c$ we proceed as follows (actually, our argument applies for all $\beta \geq 0$). We first show that $\left| \frac{\log Z_{N+1}}{N+1} - \frac{\log Z_N}{N} \right| \to 0$ as $N \to \infty$. The result then follows by use of (3.6). Now

$$\left| \frac{N \log Z_{N+1} - N \log Z_N - \log Z_N}{N(N+1)} \right| \leq \left| \frac{\log Z_{N+1}/Z_N}{N+1} \right| + \frac{1}{N+1} \left| \frac{\log Z_N}{N} \right|,$$

and we see by (3.6) and the existence of $f(\beta, 0)$ that the second term $\frac{1}{N+1} \left| \frac{\log Z_N}{N} \right| \leq \frac{K}{N+1}$ for some finite constant K. In the Appendix we show that $2^{-\beta} e^{-\beta|h|} \leq \frac{Z_{N+1}}{Z_N} \leq 2e^{\beta|h|}$ which completes our proof of the existence of the free energy for all $\beta \geq 0$ and $h \in \mathbb{R}$.

We also know rigorously that

$$f(t,0) \sim c\frac{t}{\ln t} + \dots, \tag{3.8}$$

where $c > 0$, $t = \frac{\beta_c}{\beta} - 1$, for $t > 0$ (see Fig. 1). It follows that $f(t,h)$ must have at least one singularity between the regions with low and high temperatures, i.e. a phase transition from the ordered to the high-temperature phase.

Since we can not calculate $f(\beta,h)$ exactly for $\beta < \beta_c$ (except for $h = 0$ and $\beta \to \beta_c$), we use another method, in the next section, to examine the thermodynamics.

3.3 Renormalization group analysis

3.3.1 Mean field theory

In mean field theory one assumes that there is an expansion of the free energy of the form

$$f_{MF} = a + btM^2 + uM^4 - ghM + \dots \ , \tag{3.9}$$

where M is the magnetization and the "constants" a, b, u and g are weakly dependent on the reduced temperature t (defined at the end of section 3.2) and external field h. Note that $u > 0$ is required for stability, and $b > 0, g \geq 0$ in the high-temperature phase. (The possibility that $g = 0$ is ruled out below.)

Minimizing (3.9) with respect to M, one obtains the free energy and magnetization in mean field approximation. Explicitly

1. for $t > 0$ and $h \neq 0$ the magnetization

$$M_0 \sim \frac{1}{6}\left[\frac{u}{gh} + \left(\frac{2bt}{3gh}\right)^3\right]^{-\frac{1}{3}}$$

   (note the limiting cases $M_0 \sim 0$ for $h = 0$ and $M_0 \sim 1/6(gh/u)^{1/3}$ for $t = 0$)

2. for $t < 0$ and $h \neq 0$, but h sufficiently small, the magnetization

$$M_0 \sim \left(\frac{b|t|}{2u}\right)^{\frac{1}{2}} + \frac{gh}{4b|t|}$$

(however when $\left(\frac{gh}{2u}\right)^2 + 4\left(\frac{bt}{6u}\right)^3 > 0$, M_0 is given by the $t > 0$ formula). We include this second case only for completeness. Since our system is completely saturated at low temperatures this result is not employed in our analysis.

In the following we use the first result in an RG analysis.

3.3.2 Renormalization group analysis

We assume two relevant fields (t and h) and one marginal field (u). These assumptions are reasonable, since our model has an Ising-like ordered state, the interactions are (apparently) all ferromagnetic, and there is a logarithmic term in the free energy.

The infinitesimal renormalization group transformation for the singular part of the free energy is

$$f_s(t, h, u) = e^{-d\ell} f_s(t(\ell), h(\ell), u(\ell)). \tag{3.10}$$

Because of the marginal field u, the analysis is somewhat more complicated than otherwise. We follow the treatment of Cardy (Cardy, 1996) (see also Wegner (Wegner and Riedel, 1973)). The RG equations take the form

$$du/d\ell = -xu^2 + \dots \tag{3.11}$$

$$dt/dl = y_t t - z_t u t + \dots \tag{3.12}$$

$$dh/dl = y_h h - z_h u h + \dots, \tag{3.13}$$

where we keep only the most important terms. The omitted terms are either higher order or go to zero more rapidly with ℓ than those included. From (3.11) we find (note $t = t(0)$, $h = h(0)$, $u = u(0)$)

$$u(\ell) = \frac{u(0)}{1 + xu(0)\ell}. \tag{3.14}$$

Both t and h have the same functional form, namely

$$\ln(t(\ell_0)/t(0)) = y_t \ell_0 - \frac{z_t}{x} \ln[1 + xu(0)\ell_0] \tag{3.15}$$

and

$$\ln(h(\ell_0)/h(0)) = y_h \ell_0 - \frac{z_h}{x} \ln[1 + xu(0)\ell_0], \tag{3.16}$$

where ℓ_0 is such that $t(\ell_0) = O(1)$ or $h(\ell_0) = O(1)$. From (3.15) we can write

$$\ell_0 \sim \frac{1}{y_t} \ln \frac{t_0}{t} + \frac{z_t}{x y_t} \ln \left[1 + \frac{x}{y_t}\, u \ln \frac{t_0}{t} \right], \tag{3.17}$$

where we assume $t_0/t \gg 1$. This result together with (3.10) gives us

$$f_s(t, h, u) \sim \left| \frac{t}{t_0} \right|^{\frac{d}{y_t}} \left[1 + \frac{x}{y_t}\, u \ln \frac{t_0}{t} \right]^{-\frac{z_t}{y_t}\frac{d}{x}} f_s(t(\ell_0), h(\ell_0), u(\ell_0)). \tag{3.18}$$

Since the free energy on the rhs is evaluated at ℓ_0, which is far from the critical point, it can be calculated from mean field theory. Above the critical temperature ($t > 0$) with small external field h ($h(\ell_0) \ll t(\ell_0)$) we obtain for the free energy

$$f_s(t(\ell_0), h(\ell_0), u(\ell_0)) \sim a - \frac{3(gh(\ell_0))^2}{16bt(\ell_0)}. \tag{3.19}$$

The relation between $h(\ell_0)$ and $t(\ell_0)$ follows from (3.15) and (3.16). Eliminating $h(\ell_0)$ allows us to rewrite (3.19) as

$$f_s \sim a - \left| \frac{t_0}{t} \right|^{2\frac{y_h}{y_t}} h^2 \left[1 + \frac{x}{y_t}\, u \ln \frac{t_0}{t} \right]^{2y_h\left[\frac{z_t}{y_t}\frac{}{x} - \frac{z_h}{y_h}\frac{}{x}\right]} \left(-\frac{3g^2}{16bt(\ell_0)} \right). \tag{3.20}$$

Substituting the result into (3.10) gives two terms,

$$\left| \frac{t}{t_0} \right|^{\frac{d}{y_t}} \left[1 + \frac{x}{y_t}\, u \ln \frac{t_0}{t} \right]^{-\frac{z_t}{y_t}\frac{d}{x}} a, \tag{3.21}$$

and

$$\left| \frac{t}{t_0} \right|^{\frac{d}{y_t} - 2\frac{y_h}{y_t}} h^2 \left[1 + \frac{x}{y_t}\, u \ln \frac{t_0}{t} \right]^{-\frac{z_t}{y_t}\frac{d}{x} + 2y_h\left[\frac{z_t}{y_t}\frac{}{x} - \frac{z_h}{y_h}\frac{}{x}\right]} \left(-\frac{3g^2}{16bt(\ell_0)} \right). \tag{3.22}$$

The first term can be compared with the exact result at $h = 0$ (see (3.8)). It follows that

$$\frac{d}{y_t} = 1 = \frac{z_t}{x}. \tag{3.23}$$

The second term gives us the dependence on external field. Eliminating $t(\ell_0)$ instead of $h(\ell_0)$ we obtain

$$\frac{1}{t} \left| \frac{h}{h_0} \right|^{\frac{d}{y_h} + \frac{y_t}{y_h}} \left[1 + \frac{x}{y_t}\, u \ln \frac{t_0}{t} \right]^{-\frac{z_h}{y_h}\frac{d}{x} - y_t\left[\frac{z_h}{y_h}\frac{}{x} - \frac{z_t}{y_t}\frac{}{x}\right]} \left(-\frac{3(gh(\ell_0))^2}{16b} \right). \tag{3.24}$$

Equating the two expressions (3.22) and (3.24) for the same term in the free energy gives us the RG eigenvalues

$$\frac{d}{y_t} = \frac{d}{y_h} = 1, \tag{3.25}$$

where d is the dimensionality of the system. This is of course one for our model, but since none of our results require setting $d = 1$ we leave it unspecified.

Finally we can write down the singular part of the free energy for the high-temperature phase

$$f_s(t,h,u) \sim \left|\frac{t}{t_0}\right| \left[\frac{x}{y_t}\, u \ln \frac{t_0}{t}\right]^{-1} a - \frac{h^2}{t} \left[\frac{x}{y_t}\, u \ln \frac{t_0}{t}\right]^{1-\frac{z_h}{x}} \left(\frac{3g^2}{16b}\right). \tag{3.26}$$

Since $f < 0$ for $h = 0$ in this phase, (3.26) implies that $a < 0$.

For the ordered phase we know rigorously that the free energy has no temperature dependence for $h = 0$. The spins are all up or all down. When we add an external field it will break the symmetry and all the spins will be oriented in the field direction. Thus the free energy at ℓ_0 is

$$f_s(t(\ell_0), h(\ell_0), u(\ell_0)) = -|h(\ell_0)|. \tag{3.27}$$

Proceeding as in the derivation of (3.18) from (3.10) and (3.15) we get

$$f_s(t,h,u) \sim \left|\frac{h}{h_0}\right|^{\frac{d}{y_h}} \left[1 + \frac{x}{y_h}\, u \ln \frac{h_0}{h}\right]^{-\frac{z_h}{y_h}\frac{d}{x}} f_s(t(\ell_0), h(\ell_0), u(\ell_0)), \tag{3.28}$$

using (3.16), (3.27) and (3.25) then give

$$f_s(t,h,u) \sim -|h| \left[1 + \frac{x}{y_h}\, u \ln \frac{h_0}{h}\right]^{-\frac{z_h}{x}}. \tag{3.29}$$

Because the magnetization in the ordered state is completely saturated the logarithmic correction must vanish. Therefore $z_h = 0$.

Thus the asymptotic form for the free energy of the high-temperature state is

$$f_s(t,h,u) \sim \left|\frac{t}{t_0}\right| \left[\frac{x}{y_t}\, u \ln \frac{t_0}{t}\right]^{-1} a - \frac{h^2}{t} \left[\frac{x}{y_t}\, u \ln \frac{t_0}{t}\right] \left(\frac{3g^2}{16b}\right). \tag{3.30}$$

We can recast this result more suggestively as

$$f_s(t,h,u) \sim f_s(t,0,u) - \frac{1}{2}h^2\chi(t,0,u), \tag{3.31}$$

where $\chi = -\partial^2 f/\partial h^2$ is the susceptibility. Note that $\chi \sim 1/f_s$ which is consistent with scaling theory, since (using (3.25)), $f \sim t^{2-\alpha} = t^{d/y_t} = t$ while $\chi \sim t^{-\gamma} = t^{(d-2y_h)/y_t} = t^{-1}$. This relation holds regardless of whether we set the dimensionality $d = 1$ or not. In addition, the coefficient of $\frac{t}{\ln t}$ for the free energy at $h = 0$ and $t \to 0$, $t > 0$ is known exactly (Prellberg, 1991; Prellberg and Slawny, 1992), so that the combination of constants $\frac{y_t a}{|t_0| x u}$ may be determined.

The phase boundary is given by the continuity of the free energy. Now we expect the ordered phase to exist for $\beta < \beta_c$ if h is large enough (this is reflected in the assumption of two relevant fields-if another phase intervened there would be more). Thus one must equate the two expressions for f. One finds that the phase boundary between the ordered and high-temperature phase, close to the critical point, follows

$$|h| \sim k\frac{t}{\ln t/t_0}, \tag{3.32}$$

where $k = \left\{\frac{8by_t}{3xug^2}\left[1 - \sqrt{1 + \frac{3ag^2}{4bt_0}}\,\right]\right\}$. Since f is quadratic in h in the high-temperature phase, there are in general two solutions with $h > 0$. However, the one at larger h is not physical since it gives rise to a magnetization $m > 1$ and violates the convexity of the free energy as well, so we employ the other. In order to find the change in magnetization across the phase boundary we use (3.32) with constants included

$$|h| \sim \frac{-t}{\ln \frac{t_0}{t}}\left\{\frac{8by_t}{3xug^2}\left[1 - \sqrt{1 + \frac{3ag^2}{4bt_0}}\,\right]\right\}. \tag{3.33}$$

In arriving at (3.33), we (as mentioned) chose the root that makes $m < 1$ in the high-temperature phase. Note that in the limiting case that $\frac{3ag^2}{4bt_0} = -1$, $m = 1$ but the two roots coincide.

Now from (3.30)

$$m \sim \frac{h}{t}\left[\frac{x}{y_t}\, u \ln\frac{t_0}{t}\right]\left(\frac{3g^2}{8b}\right). \tag{3.34}$$

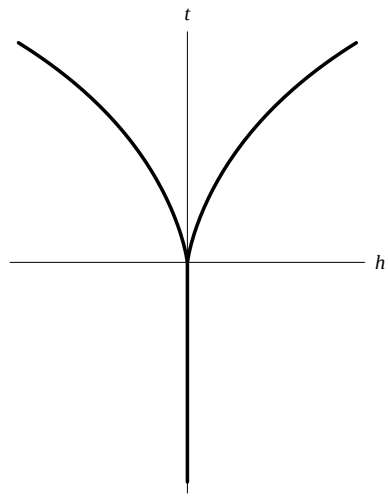

Figure 3: Phase diagram

Eliminating the external field using (3.33), and since the magnetization in the ordered phase takes the values $m \sim \pm 1$, we find

$$\Delta m \sim \sqrt{1 + \frac{3ag^2}{4bt_0}}. \tag{3.35}$$

Note that t_0 is a constant of order one and recall that $a < 0$, thus on the phase boundary the discontinuity in magnetization is constant (and less than one), at least close to the second-order point (we argue below that $g = 0$ is not possible in this model). Now we can look at the change in entropy (per site) $s = \beta^2 \, \partial f / \partial \beta$ across the phase boundary. We get

$$\Delta s \sim -2 \left[\frac{x}{y_t} \, u \ln \frac{t_0}{t} \right]^{-1} \left(\frac{a}{t_0} + \frac{4b}{3g^2} \left[1 - \sqrt{1 + \frac{3ag^2}{4bt_0}} \right] \right). \tag{3.36}$$

These results show that the phase transition is first-order everywhere except at $h = 0$.

In the limiting case when $\frac{3ag^2}{4bt_0} = -1$, already mentioned, one finds that both $\Delta m = 0$ and $\Delta s = 0$. However, it is easy to see that both the susceptibility χ and the specific heat will have a discontinuity across the phase boundary.

Note that the magnetization change given by (3.35) exhibits a kind of "discontinuity of the discontinuity", in that its limiting value as one approaches the second-order

point is not the same as its value at that point. This is not the case for the entropy change, or for these quantities in the model examined in section 3.4.

Finally, we argue that $g = 0$ is not possible in the high-temperature phase. Since the second derivative of f with respect to h at $h = 0$ is proportional to both g and the susceptibility χ, it suffices to demonstrate that $\chi > 0$. It is straightforward to show that χ is proportional to $\Sigma_{j=1}^{N}\langle s_1 s_j\rangle$ where the spin variables $s_i := 2\sigma_i - 1, \quad s_i \in \{-1, 1\}$ (cf. (3.1)), and the angular brackets denote a thermal average. Now the $j = 1$ term in this sum is 1, and due to the ferromagnetic interactions in the spin chain, the remaining terms cannot be negative. Note that this argument is not completely rigorous, since for the FFSC we only have numerical evidence that the interactions are all ferromagnetic. The KSC, on the other hand, is known to have all interactions ferromagnetic (Contucci et al., 1999), so that $\langle s_1 s_j\rangle > 0$ follows from the GKS inequalities.

3.3.3 Finite-size scaling

We can use our results to make some predictions about finite-size (i.e. N large but $N < \infty$) effects on the thermodynamics. We make the standard assumption that the size of our spin chain is a relevant field with eigenvalue 1. Of course, since our system has long-range interactions the validity of finite-size scaling may be questioned (Cardy, 1996), but it is still interesting to see the results. The treatment is the same as in the case of the relevant fields t and h. The renormalization equation for the inverse size $I := N^{-1}$ is then

$$dI/dl = I - z_I u I + \dots \ . \tag{3.37}$$

Thus we get

$$f_s(t, h, u, N^{-1}) \sim \left|\frac{N_0}{N}\right|^d \left[1 + x\ u \ln \frac{N}{N_0}\right]^{-\frac{z_I\, d}{x}} f_s(t(\ell_0), h(\ell_0), u(\ell_0), N^{-1}(\ell_0)). \tag{3.38}$$

Note that we do not know the ratio z_I/x, however (3.38) gives the form we should observe. More succinctly, for large N, this result predicts that for small t and h

$$\ln Z_N(t,h) \sim N^{1-d}[\ln N]^{-p}. \tag{3.39}$$

There is related work in number theory by Kanemitsu (Kanemitsu, 1996) (cf. also (Shigeru et al., 2000)). This paper studies moments of neighboring Farey fraction differences, which are similar to the "Farey tree" partition function (Feigenbaum et al., 1989). At $h = 0$, the latter has the same thermodynamics as the FFSC (Chapter 2 and (Fiala et al., 2003). However, (Kanemitsu, 1996) uses a definition of the Farey fractions that, at each level, gives a subset of the Farey fractions employed here, and none of the moments considered corresponds to $\beta = 2$ (the point of phase transition). It is interesting that, despite these differences, terms logarithmic in N appear. More specifically, the sum of mth (integral) moments of the differences goes as

$$\frac{(\ln N)^{\delta_{2,m}}}{N^m} + O\left(\frac{(\ln N)^{h(m)}}{N^{m+g(m)}}\right), \tag{3.40}$$

for $m \geq 2$, with $g(2) = 1$, $g(3) = 2$ and $g(m) = 3$ for $m \geq 4$, and $h(m) = 1$ for $2 \leq m \leq 4$, $h(m) = 0$ for $m \geq 5$. Now if all the Farey fractions were included (3.40) would apply to the Farey tree partition function with $\beta = 2\,m$ (Chapter 2 (Fiala et al., 2003; Feigenbaum et al., 1989)) so that $m \geq 2$ would correspond to $\beta \geq 4$. It would be interesting to determine whether (3.40) applies to the Farey tree partition function despite this difference, or to extend (3.40) to $m = 1$ to see if it is consistent with (3.39).

3.4 1-D KDP model with nonzero external field

In this section we consider the one-dimensional KDP (Potassium dihydrogen phosphate) model introduced by Nagle (Nagle, 1968). This model's thermodynamics and

energy level structure are similar to the Farey fraction spin chain, but it is easily solvable. Comparison of the two models thus sheds some light on the FFSC.

The KDP model exhibits first-order phase transitions only. The origin of the phase transition is infinite rather than long-range interactions. The one-dimensional

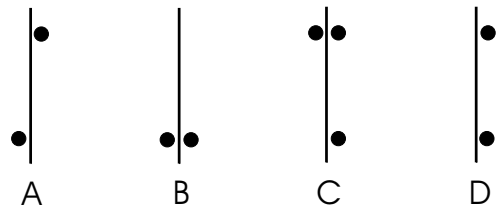

Figure 4: KDP

geometry of the model is illustrated in Fig. 4. It consists of N cells, and each cell contains two dots. Each dot represents a proton in a hydrogen bond in the KDP molecule. Dots can be on the left or the right side of a cell. The energy of a neighboring pair of cells depends on the arrangement of dots at their common boundary. Only configurations with exactly two dots at each boundary (e.g. A, B and D in Fig. 4) are allowed, any other configuration (e.g. C in Fig. 4) has (positively) infinite energy and is therefore omitted. Of the allowed configurations, only two energies occur, 0 (when there are two dots on the same side of a boundary, as in Fig. 4 D) or ϵ (when the dots are on opposite sides, as in Fig. 4 A or B).

Let there be N cells in a chain with periodic boundary conditions. Then there are two kinds of configurations with finite energy. In the first type of configuration, each cell has two dots on the same side. There are two such configurations and the total energy of each is 0. In the second type of configuration, each cell has one dot on the left and one on the right. There are 2^N such configurations and the total energy of each is $N\epsilon$. Thus, the partition function is simply

$$Z_N(\beta) = 2 + 2^N \exp(-\beta N \epsilon). \tag{3.41}$$

It follows immediately that $f = 0$ for $\beta\epsilon > \ln 2$ and $f = \epsilon - \frac{\ln 2}{\beta}$ for $\beta\epsilon < \ln 2$. Thus the temperature of the (first-order) phase transition is $T_c = \epsilon/(\ln 2)$ and there is a

latent heat with entropy change $\Delta s = \ln 2$. Clearly, the phase transition mechanism is a simple entropy-energy balance. At low temperatures, the ground state energy gives the minimal free energy, while in the high-temperature phase the extra entropy of the additional states gives a lower free energy.

Next, define the magnetization m as the number of sides of cells with both dots on one side divided by the number of cells N. Then $m = 1$ for $\beta > \beta_c$ and $m = 0$ for $\beta < \beta_c$ (so that $\Delta m = 1$ at the phase transition), just as in the FFSC model.

Following the above definition of the magnetization, we introduce an external field h by adding an energy $\pm h/2$ to each dot, according to whether it is on the right or left side of the cell. This gives the extra energy of an external field acting along the chain. Then the new partition function has the form

$$Z_N(\beta, h) = 2\cosh(\beta N h) + 2^N \exp(-\beta N \epsilon). \tag{3.42}$$

In the ordered phase $Z \to \exp(\pm\beta N h)$. Thus, the free energy $f = \mp h$, where the plus sign is for $h > 0$ and the minus sign for $h < 0$, exactly as in the FFSC. For the high-temperature phase $Z \to \exp\Big[N(\ln 2 - \beta\epsilon)\Big]$ and we get the same free energy as when $h = 0$, $f = \epsilon - \frac{\ln 2}{\beta}$. The phase boundary is given by $h = \pm\epsilon t$ (see Fig. 5), where $t = \frac{\beta_c}{\beta} - 1$ as before. Note the resemblance to the FFSC phase diagram (Fig. 3). Here, as $\beta\epsilon \to \ln 2$, $h \to 0$ as it should, while for $\beta \to 0$ the field $h \to \ln 2/\beta$. The entropy per site vanishes everywhere in the ordered phase, while for the high-temperature phase $s = \ln 2$. Thus, this model has a non-zero latent heat and the phase transition is first-order everywhere. Note that the change in magnetization is $\Delta m = 1$ everywhere along the phase boundary between the ordered state and the high-temperature state. Now for $h = 0$, the FFSC has two ground states with all spins up or all spins down and energy independent of length N, just as in the KDP model. Then, in addition, the FFSC has $2^N - 2$ states with energies between $\ln N$ and Nc, for some constant c. On the other hand, the KDP model has just one energy ($N\epsilon$) for the 2^N states corresponding to the $2^N - 2$ states of the Farey model.

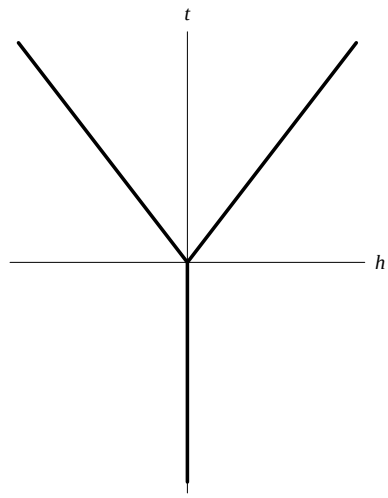

Figure 5: Phase diagram

This might suggest that the states with energies close to $\ln N$ are responsible for the logarithmic factor in the Farey free energy, and thus shift the phase transition from first to second-order (for $h = 0$). For $h \neq 0$ the energy of the $\ln N$ states is shifted by the field h to order N, and the phase transition becomes first-order. However the mechanism of the FFSC phase transition may be more subtle. The "density of states" (number of configurations with a given energy) for the FFSC not well-behaved. In fact it is known rigorously that this quantity, summed over all chain lengths, has a limit distribution (Peter, 2001).

Note that the free energy just derived is independent of h in the high-temperature phase. Since this is not what we found for the FFSC, we consider another way to introduce an external field h into the KDP model. As before we have four different states for each cell. We index them with spin-one variables t_i and s_i ($s_i,\ t_i \in \{0, +1, -1\}$) in each cell as in Fig. 6. Then the energy (for $h = 0$) can be written

$$H_0 = \epsilon \sum_{i=1}^{N-1} t_i^2 t_{i+1}^2 \tag{3.43}$$

(assuming, in the sum, that the infinite energy contributions are omitted). The conditions $s_i + t_i = \pm 1$ and $s_i t_i = 0$ define the allowed states. We define the magnetization

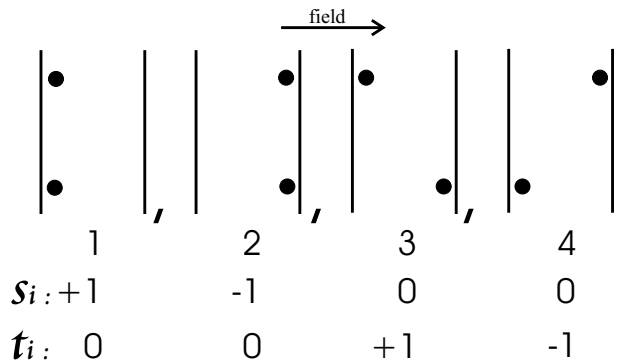

Figure 6: Notation

per site as

$$m = \frac{1}{N}\sum_{i=1}^{N}(s_i + t_i). \tag{3.44}$$

Note that this definition gives a positive (negative) contribution if the upper dot in a given cell is on the right (left). (Note also that $m^2 = \frac{1}{N^2}\sum_{i=1}^{N}(s_i+t_i)^2 + \frac{1}{N^2}\sum_{i\neq j}(s_i+t_i)(s_j+t_j) = \frac{1}{N}\sum_{j=2}^{N}(s_1+t_1)(s_j+t_j) + 1/N$.) Hence we can include an external field as follows

$$H = H_0 - h\sum_{i=1}(s_i + t_i) = H_0 - hNm. \tag{3.45}$$

Thus

$$Z(\beta, h) = e^{\beta N h} + e^{-\beta N h} + e^{-\beta\epsilon N}[2\cosh(\beta h)]^N, \tag{3.46}$$

and the free energy in high-temperature phase becomes

$$f(\beta, h) = \epsilon - \frac{\ln(2\cosh(\beta h))}{\beta} \tag{3.47}$$

or for small h

$$f \sim -t\epsilon - \frac{\ln 2}{2\epsilon(t+1)}h^2, \tag{3.48}$$

with $t = \frac{\beta_c}{\beta} - 1$ as above. The phase boundary is given by

$$\beta h = \ln\Big(2\cosh(\beta h)\Big) - \beta\epsilon. \tag{3.49}$$

For $\beta h \ll 1$ and $h > 0$, using $\beta_c = \frac{\ln 2}{\epsilon}$, this gives

$$h = \epsilon\, t + \frac{\epsilon \ln 2}{2}\, t^2 + O(t^3), \tag{3.50}$$

The phase diagram near the critical point is very close to the previous one (see Fig. 5). The magnetization in the ordered phase is again independent of temperature, i.e. $m = \pm 1$. In the high-temperature phase we have $m = \tanh(\beta h)$. Thus the magnetization change across the phase boundary close to the critical point is $\Delta m = 1 - t \ln 2$. The transition is again first-order, with the entropy change $\Delta s = \ln 2(1 - \frac{\ln 2}{2} t^2)$. Results for $h < 0$ follow immediately by symmetry.

CHAPTER 4

TRANSFER OPERATOR, EXPECTATION VALUES, AND CORRELATION FUNCTIONS

In this chapter we extend the definition of Knauf's "number theoretical" partition function by introducing a new parameter x. This allows us to write recurrence relations for the partition functions. These relations are shown to imply a simple and direct connection between the operator studied by Prellberg (see Chapter 2), and the transfer operator of Contucci and Knauf. We examine the consequences of this connection. In addition, the recurrence relations allow us to calculate certain spin expectation values and correlation functions.

4.1 Definition of the partition function

Let the matrix M_k be any product of k matrices $A_0 := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $A_1 := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$,

$$M_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we can extend the Knauf model (Knauf, 1998) of eq. (2.2) (see also (Kleban and Özlük, 1999)) by introducing a family of partition functions parametrized by the variable x (Zagier, 2002)

$$\tilde{Z}_k(x, \beta) := \sum (cx + d)^{-2\beta}, \tag{4.1}$$

where the sum runs over all 2^k permutations of the product of the k matrices A_0, A_1. We show in section 4.3 that all of the partition functions (4.1) have the same free energy.

Now consider the partition function of the length $k+1$ (i.e. $M_{k+1} = M_k A_0 + M_k A_1$, where $M_k A_0 = \left(\begin{smallmatrix} a+b & b \\ c+d & d \end{smallmatrix}\right)$ and $M_k A_1 = \left(\begin{smallmatrix} a & a+b \\ c & c+d \end{smallmatrix}\right)$). From (4.1) we find the recurrence equation

$$\tilde{Z}_{k+1}(x,\beta) = (1+x)^{-2\beta} \tilde{Z}_k \left(\frac{x}{1+x}, \beta \right) + \tilde{Z}_k(x+1,\beta) \tag{4.2}$$

with the initial condition $\tilde{Z}_0(x,\beta) \equiv 1$ (i.e. $M_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$). The variable $x \in \mathbb{R}_0^+$ is a parameter which changes the energy of each configuration and $\beta \in \mathbb{R}_0^+$ is the inverse temperature.

It is convenient to define, as in number theory (Zagier in (Waldschmidt et al., 1992)) the action of the matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on any function $f(x)$

$$f(x)|M := (cx+d)^{-2\beta} f\left(\frac{ax+b}{cx+d} \right). \tag{4.3}$$

For example, consider the action of the matrix A_0 on a constant function

$$1(x)|A_0 = (1+x)^{-2\beta}, \tag{4.4}$$

where $1(x) \equiv 1$.

It is easy to check that our partition function $\tilde{Z}_k(x,\beta)$ can be written as

$$\tilde{Z}_k(x,\beta) = \sum_{i=1}^{2^k} 1(x)|M_i, \tag{4.5}$$

where $M_i = \Pi_{j=1}^k A_{\tau_j(i)}$ with $\tau_j(i) \in \{0,1\}$. The matrices $A_0 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $A_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ can be viewed as a spin up or spin down, respectively. Note that each product of these two matrices defines parent fractions of the fraction in the level k of the Stern-Brocot tree (Graham et al., 1994). The subset of these fractions between zero and one are called Farey fractions. They are generated by the products which start with A_0 (Kleban and Özlük, 1999).

In the following we will use an abbreviated form of (4.5)

$$\tilde{Z}_k(x,\beta) = 1(x)|(A_0+A_1)^k = 1(x)|A_0(A_0+A_1)^{k-1} + 1(x)|A_1(A_0+A_1)^{k-1}, \tag{4.6}$$

where the addition must be applied **after** the multiplication of the matrices!

The Knauf "canonical" partition function $Z_k^K(s)$ (see (Knauf, 1998)) the definition and note that $s = 2\beta$) is equal to

$$Z_k^K(2\beta) = 1(x)|A_0(A_0 + A_1)^k|_{x=0}. \tag{4.7}$$

Similarly, the grand canonical partition function of Contucci and Knauf (Contucci and Knauf, 1997) corresponds to (4.7) with $x = 1$ on the right hand side.

Let

$$Z_k(x, \beta) := 1(x)|A_0(A_0 + A_1)^k. \tag{4.8}$$

(4.8) is than a direct generalization of Z_k^K. Then using $1(x)|A_1 = 1(x)$ and (4.6) we get

$$\tilde{Z}_k(x, \beta) = Z_{k-1}(x, \beta) + 1(x)|A_1(A_0 + A_1)^{k-1} = 1 + \sum_{i=0}^{k-1} Z_i(x, \beta). \tag{4.9}$$

Thus (4.9) relates two partition functions satisfying the recurrence formula (4.2) with initial conditions $Z_0(x, \beta) = (1 + x)^{-2\beta}$ and $\tilde{Z}_0(x, \beta) = 1(x)$. In the next section we will study the invariance of $Z_0(x, \beta)$ under the action of the matrix $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (Pauli matrix).

4.2 Spin orientation invariance and their consequences

In this section we consider the consequences of the spin flip transformation generated by $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In addition we observe that there is subset of the partition function which exhibits simple transformation property and the rest of partition functions are directly connected to this special subset.

The action of P on a function $f(x)$ is

$$f(x)|P = x^{-2\beta} f(1/x). \tag{4.10}$$

In our case, the matrix P simply exchanges the spin orientation, i.e. the matrix A_0 and the matrix A_1

$$A_1 = P\, A_0\, P, \tag{4.11}$$

where $P^2 = 1$. Thus A_0 and A_1 are conjugate. Note that a function $f(x)$ invariant under (4.10) (i.e. $f(x) = x^{-2\beta} f(1/x)$) can be called even, since, using the substitution $e^y = x$ (x is non-negative) to define $g(y) = e^\beta f(e^y)$, (4.10) becomes $g(y) = g(-y)$. Thus our initial condition $(1+x)^{-2\beta}$ is even. Consequently, for all for all $k \geq 1$, $x \in \mathbb{R}^+$ and $\beta \in \mathbb{R}_0^+$ the partition function $Z_k(x)$ is even

$$Z_k(x)|P = (1+x)^{-2\beta}|(A_0 + A_1)^k P = (1+x)^{-2\beta}|P^2(A_0 + A_1)^k = Z_k(x). \tag{4.12}$$

In the second equality we used the evenness of our initial condition and the fact that the set of all terms in $(A_0 + A_1)^k P$ is the same as the set $P(A_0 + A_1)^k$.

Now consider the terms in (4.2). Using the invariance property of our partition function we can write $Z_{k-1}(x)|A_0 = Z_{k-1}(x)|P\,A_0$ and $Z_{k-1}(x)|A_1 = Z_{k-1}(x)|P\,A_1$. Thus

$$(1+x)^{-2\beta} Z_{k-1}\left(\frac{x}{1+x}, \beta\right) = x^{-2\beta} Z_{k-1}\left(\frac{1+x}{x}, \beta\right) \tag{4.13}$$

and

$$Z_{k-1}(x+1, \beta) = (1+x)^{-2\beta} Z_{k-1}\left(\frac{1}{1+x}, \beta\right) \tag{4.14}$$

for all $k \geq 1$, $x \in \mathbb{R}^+$ and $\beta \in \mathbb{R}_0^+$. Combining (4.13), (4.14) and (4.2) gives us four different recurrence formulas. For instance

$$Z_k(x) = (x+1)^{-2\beta}\left[Z_{k-1}\left(\frac{x}{x+1}\right) + Z_{k-1}\left(\frac{1}{x+1}\right)\right] \tag{4.15}$$

which we will use in section 4.3. In addition we can see that the matrix P can be put in front of the matrix A_0 or A_1 in the expression $(1+x)^{-2\beta}|(A_0+A_1)^k$ and not change the partition function $Z_k(x)$ (for example $(1+x)^{-2\beta}|(A_0+A_1)^l(P\,A_0+A_1)(A_0+A_1)^r = (1+x)^{-2\beta}|(A_0+A_1)^k$ for any $k, l, r \geq 0$ such that $l + r + 1 = k$). On the other hand if we put the matrix P after only the matrix A_0 or A_1 we get a new function. Let

$$Z_k^{l\uparrow r}(x) = \frac{1}{2}(1+x)^{-2\beta}|(A_0 + A_1)^l(A_0 + A_1 P)(A_0 + A_1)^r \tag{4.16}$$

and

$$Z_k^{l\downarrow r}(x) = \frac{1}{2}(1+x)^{-2\beta}|(A_0 + A_1)^l(A_0 P + A_1)(A_0 + A_1)^r, \tag{4.17}$$

with $l + r + 1 = k$. Using (4.11) we then have

$$Z_k^{l\uparrow r}(x) = Z_l(x)|A_0(A_0 + A_1)^r \tag{4.18}$$

and

$$Z_k^{l\downarrow r}(x) = Z_l(x)|A_1(A_0 + A_1)^r \tag{4.19}$$

respectively, which motivates the notation in (4.16) and (4.17). The arrows $\uparrow$ and $\downarrow$ refer to the interpretation of A_0 and A_1, as up and down spins, respectively. In addition note that

$$Z_k^{l\uparrow r}(x) + Z_k^{l\downarrow r}(x) = Z_k(x). \tag{4.20}$$

We will study these functions in section 4.4. We conclude with an observation which follows immediately from (4.18) and (4.19). The probability of a spin up at position $l+1$ from left is equal to the probability of a spin down at the same position for model with different x. The relation is

$$\frac{Z_k^{l\uparrow r}(x)}{Z_k(x)} = \frac{Z_k^{l\downarrow r}(x)|P}{Z_k(x)} = \frac{x^{-2\beta} Z_k^{l\downarrow r}(1/x)}{Z_k(x)|P} = \frac{Z_k^{l\downarrow r}(1/x)}{Z_k(1/x)} \tag{4.21}$$

for all for all $k \geq 1$, $x \in \mathbb{R}^+$ and $\beta \in \mathbb{R}_0^+$. Note that for $x = 1$ these probabilities are equal. For other values of x, since the magnetization is zero, the up and down spins probabilities are equal when l and r are sent to infinity (see section 4.4).

4.3 Connection to the transfer operator

In Section 2.3 we defined the transfer operator on the Farey map (see (2.21) and (2.22)). The transfer operator is formally given by

$$\mathcal{K}_\beta\, \varphi(x) \quad = \quad |F_0{}'(x)|^\beta \varphi(F_0(x)) + |F_1{}'(x)|^\beta \varphi(F_1(x)). \tag{4.22}$$

Therefore, the k-fold iterated operator $\mathcal{K}_\beta\, \varphi(x)$ consists of 2^k terms of the form

$$|(F_{\tau_1} \circ F_{\tau_2} \circ \ldots \circ F_{\tau_k})'(x)|^\beta \varphi(F_{\tau_1} \circ F_{\tau_2} \circ \ldots \circ F_{\tau_k}(x)) \tag{4.23}$$

with $\tau_j \in \{0,1\}$. As we are dealing with iterations of Möbius transformations of the form $\frac{ax+b}{cx+d}$ with determinant ± 1, we can alternatively consider multiplication of the associated matrices. We find for instance

$$\mathcal{K}_\beta^k 1(x) = \sum_{\{\tau_j\}} (cx+d)^{-2\beta}_{\{\tau_j\}} = \sum_{i=1}^{2^k} 1(x)|\tilde{M}_i, \tag{4.24}$$

where c and d are just the bottom left and right entries, respectively,

$$\tilde{M}_i = \prod_{j=1}^{k} F_{\tau_j(i)} \quad \text{where} \quad F_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} . \tag{4.25}$$

Note that $A_0 = F_0$ and $F_1 = P\,A_1$.

When we apply $\mathcal{K}_\beta$ only once on the constant function $1(x)$ we obtain $2(1+x)^{-2\beta}$. That is exactly twice the initial condition of the partition function $Z_k(x)$ (see (4.8)). In addition $\mathcal{K}_\beta$ increases the level k of the partition function $Z_k(x)$ by one as follows from (4.15 and 4.22). Thus

$$\mathcal{K}_\beta^k 1(x) = 2\mathcal{K}_\beta^{k-1}(1+x)^{-2\beta} = 2Z_{k-1}(x). \tag{4.26}$$

For $x = 0$, (4.26) connects the Knauf model (4.7) (Knauf, 1998) and the transfer operator $\mathcal{K}_\beta$

$$\mathcal{K}_\beta^k 1(x)|_{x=0} = 2\,Z_{k-1}^K(2\beta). \tag{4.27}$$

Note that (4.27) simplifies and extends the results of Contucci and Knauf (Knauf, 1998). These authors define an operator $\tilde{\mathcal{C}}(2\beta)$ (see Section 2.3) whose non-degenerate leading eigenvalue gives the free energy of "grand canonical" partition function $Z_k(x = 1, \beta)$ and the canonical case $Z_k(x = 0, \beta)$ as in (4.34) below. These authors also connect the largest eigenvalue of $\tilde{\mathcal{C}}(2\beta)$ with the largest eigenvalue of the equation

$$\lambda(\beta)f(x) = f(x+1) + x^{-2\beta} f(1+1/x), \tag{4.28}$$

and in fact their proof uses a Taylor series expansion of $\varphi(x)$ (in (4.22)) at $x = 1$ and it can be shown that their operator has the same spectrum as (4.22) on L^2 space.

However, their result does not include direct connection of the partition functions and (4.22) or (4.28).

It is also interesting to consider (4.27) for $\beta > \beta_c = 1$. In that case, one has

$$\lim_{k\to\infty} \mathcal{K}_\beta^k 1(x)|_{x=0} = \lim_{k\to\infty} 2\, Z_{k-1}^K(2\beta) = 2\frac{\zeta(2\beta-1)}{\zeta(2\beta)}, \tag{4.29}$$

where ζ is the Riemann-ζ function (Knauf, 1993).

For $\beta < \beta_c = 1$, the leading eigenvalue $\lambda(\beta) > 1$ of $\mathcal{K}_\beta$ (Prellberg, 2003) is non-degenerate and belongs to the discrete spectrum. Thus we can define $a(x,\beta)$ as

$$a(x,\beta) = \lim_{k\to\infty} \frac{Z_k(x,\beta)}{\lambda^k(\beta)} < \infty. \tag{4.30}$$

Note that since the spectrum of $\mathcal{K}_\beta$ is independent of x, the free energy

$$f(\beta) := \frac{-1}{\beta} \lim_{k\to\infty} \frac{\ln Z_k(x,\beta)}{k} \tag{4.31}$$

depends only on the inverse temperature $\beta < \beta_c$ (Chapter 2 and (Fiala et al., 2003; Knauf, 1998). Thus (as we have already noted for $x = 0$ in Chapter 2 and (Fiala et al., 2003)) the phase transition is second-order for all $x \geq 0$. This follows from the result of Prellberg (Prellberg and Slawny, 1992; Prellberg, 1991), already quoted in Section 2.4,

$$\beta f(\beta) = c\frac{1-\beta}{\ln(1-\beta)}[1+o(1)], \quad \beta \to 1^-, \tag{4.32}$$

where $c > 0$ (for more discussion about the phase transition see Chapter 2 and (Fiala et al., 2003)). The partition function $Z_k(x,\beta)$ can be written exactly as (4.1), except that one only sums over a subset of c and d. Thus

$$(1+(k+1)x)^{-2\beta} \leq Z_k(x,\beta) \leq Z_k(0,\beta) = Z_k^K(2\beta). \tag{4.33}$$

Since the Knauf free energy vanishes for $\beta \geq \beta_c$, so must the free energy obtained from $Z_k(x,\beta)$. Now since the leading eigenvalue of $\mathcal{K}_\beta$ is $\lambda(\beta) = 1$ for all $\beta \geq \beta_c$ we can write for all temperatures

$$f(x,\beta) = \frac{-1}{\beta} \ln \lambda(\beta). \tag{4.34}$$

Note that the leading eigenvalue changes its character at the critical point. Below the critical temperature it belongs to a discrete spectrum and above the critical temperature it is the upper limit of the continuous spectrum (for more details about the spectrum see (Prellberg, 2003)).

The sub-leading eigenvalue in the spectrum (Prellberg, 2003) is equal to one for all temperatures. This is consistent with our results Chapter 3 and (Fiala and Kleban, 2004) based on scaling and renormalization group arguments. For a one-dimensional system the scaling arguments provide the relation between singular part of free energy f_s and correlation length ξ

$$f_s \propto \frac{1}{\xi}. \tag{4.35}$$

If we assume that our partition function goes as

$$Z_k(x,\beta) = \lambda^k a(x) + \lambda_1^k a_1(x) + \ldots, \tag{4.36}$$

we obtain using (4.31)

$$f_s \propto \ln \lambda \tag{4.37}$$

and from definition of correlation length

$$\xi = \frac{C}{\ln \lambda - \ln \lambda_1} \tag{4.38}$$

where C is positive constant. We can see that this implies that the sub-leading eigenvalue $\lambda_1(\beta) = 1$ for $\beta \leq \beta_c$ consistent with Prellberg's results.

In addition, note that the eigenfunction $a(x,\beta)$ is even

$$a(x,\beta) = x^{-2\beta} a(1/x,\beta). \tag{4.39}$$

Using this fact and (4.22) we can write

$$\lambda(\beta) a(x,\beta) = a(x+1,\beta) + (1+x)^{-2\beta} a\left(\frac{x}{x+1},\beta\right). \tag{4.40}$$

Applying (4.39) and (4.40) with the $x = 0$, $x = 1$ we obtain

$$a(1,\beta) = (\lambda(\beta) - 1) a(0,\beta), \tag{4.41}$$

$$a(2,\beta) = \frac{\lambda(\beta)}{2} a(1,\beta) = \frac{\lambda(\beta)}{2}(\lambda(\beta)-1)a(0,\beta), \tag{4.42}$$

respectively. We will make extensive use of (4.41) and (4.42) below.

4.4 Expectation values and correlations

In this section we calculate various spin expectation values for finite chain and correlation functions for chains of finite length. First, consider the expectation value for spin up

$$\langle \underbrace{\cdots}_{l} \uparrow \underbrace{\cdots}_{r} \rangle_x := \frac{Z_k^{l\uparrow r}(x)}{Z_k(x)}, \tag{4.43}$$

and similarly for spin down. Now using (4.18), and (4.20) we find

$$\langle \underbrace{\cdots}_{l} \uparrow \underbrace{\cdots}_{r} \rangle_x \quad = \quad \frac{Z_l(x)|A_0(A_0+A_1)^r}{Z_l(x)|A_0(A_0+A_1)^r + Z_l(x)|A_0(A_0+A_1)^r P}. \tag{4.44}$$

The question is if we can relate the two terms in denominator at least for some values of x. We already know from (4.21) that for $x = 1$ these terms are equal. There is a simple explanation for this. First of all the P matrix in the end of every chain M_i just switches the columns of these matrices. (4.3) is invariant under change of columns for $x = 1$. Thus the probability to find spin up (or down) at any location of the spin chain with $x = 1$ is equal 1/2

$$\langle \underbrace{\cdots}_{l} \uparrow \underbrace{\cdots}_{r} \rangle_{x=1} = \langle \underbrace{\cdots}_{l} \downarrow \underbrace{\cdots}_{r} \rangle_{x=1} = \frac{1}{2} \tag{4.45}$$

Thus, in this case, there are no finite size or edge effects at all. Another quite simple result is for $x = 0$ (the Knauf model). The partition function at level k is

$$Z_k(0) = (Z_l(x)|A_0(A_0+A_1)^r + Z_l(x)|A_0(A_0+A_1)^r P)|_{x=0} = 2Z_k^{l\uparrow r}(0) + Z_l(1) - Z_l(0). \tag{4.46}$$

This result follows directly from the structure of the Farey fractions together with the action of the matrix P. Alternatively we can prove it by using (4.18), (4.19), which

we now proceed to do. First express (4.18) as the sum of one level shorter chains

$$Z_k^{l\uparrow r}(x) = (1+x)^{-2\beta} Z_{k-1}^{l\uparrow r-1}\left(\frac{x}{1+x}\right) + Z_{k-1}^{l\uparrow r-1}(x+1). \tag{4.47}$$

For $x = 0$ (4.47) becomes

$$Z_k^{l\uparrow r}(0) = Z_l(0) + \sum_{i=l+1}^{k-1} Z_i^{l\uparrow i-l-1}(1). \tag{4.48}$$

Similarly we find

$$Z_k^{l\downarrow r}(0) = Z_l(1) + \sum_{i=l+1}^{k-1} Z_i^{l\downarrow i-l-1}(1). \tag{4.49}$$

Finally we add the above expressions (see(4.20)), eliminate equal sums (see (4.21)) and (4.46) follows.

Now the expectation value for spin up at $x = 0$ can be written as

$$\langle \underbrace{\ldots}_{l} \uparrow \underbrace{\ldots}_{r} \rangle_{x=0} \quad = \quad \frac{1}{2 - \frac{Z_l(0) - Z_l(1)}{Z_k^{l\uparrow r}(0)}} = \frac{1}{2} + \frac{K}{4} + O(K^2), \tag{4.50}$$

where

$$K = \frac{Z_l(0) - Z_l(1)}{Z_l(x)|A_0(A_0 + A_1)^r|_{x=0}}. \tag{4.51}$$

The constant $K \geq 0$ is less than 1 for all $l, r \geq 0$ as follows from $Z_l(x)|A_0(A_0 + A_1)^r|_{x=0} \geq Z_l(0) \geq Z_l(1) > 0$ for all $l, r \geq 0$ and $\beta \geq 0$. The first inequality follows immediately from the fact that the sum $Z_l(x)|A_0(A_0 + A_1)^r|_{x=0}$ of positive terms includes the term $Z_l(x)|A_0^{r+1}|_{x=0} = Z_l(0)$. The second inequality follows directly from the monotonicity of our partition function $Z_l(x)$. Note that for $\beta > 0$ the partition function is a strictly decreasing function of x. Thus the spin at any position for finite temperature T and any finite length of the spin chain has greater probability to be up

$$\langle \underbrace{\ldots}_{l} \uparrow \underbrace{\ldots}_{r} \rangle_{x=0} \quad > \quad \langle \underbrace{\ldots}_{l} \downarrow \underbrace{\ldots}_{r} \rangle_{x=0}. \tag{4.52}$$

This is probably an effect of the "hidden" spin up in our initial condition $(1+x)^{-2\beta} = 1(x)|A_0$ (where the matrix A_0 represents spin up), assuming that the spin interaction are all ferromagnetic, as in the Knauf model (Knauf, 1993).

Now we consider the two-spin correlation function. Let

$$\langle \underbrace{\cdots}_{l} \uparrow \underbrace{\cdots}_{n} \uparrow \underbrace{\cdots}_{r} \rangle_x \quad = \quad \frac{Z_k^{l\uparrow n}(x)|A_0(A_0+A_1)^r}{Z_{k+r+1}(x)}, \tag{4.53}$$

where as before $k = l + n + 1$.

The partition function $Z_{k+r+1}(x)$ for a spin chain of length $l + n + r + 2$ can be divided to four terms (corresponding to four configurations of two spins)

$$Z_k^{l\uparrow n}(x)|A_i(A_0+A_1)^r$$

and

$$Z_k^{l\downarrow n}(x)|A_i(A_0+A_1)^r$$

where $i \in \{0, 1\}$. Using the matrix P (as before) gives

$$Z_k^{l\uparrow n}(x)|A_i(A_0+A_1)^r = Z_k^{l\downarrow n}(x)|A_{i+1(\mathrm{mod}\,2)}(A_0+A_1)^r P, \tag{4.54}$$

where $i \in \{0, 1\}$. We now check if we can get some results for particular values of x. As mentioned, for $x = 1$ each spin has equal probability to be up and down without any edge effect (i.e. for any l, $r \in \mathbb{Z}_0^+$). Thus we can expect the expectation value for two spins up to be the same as for two spins down. This in fact follows directly from (4.54). That result implies,

$$\langle \underbrace{\cdots}_{l} \uparrow \underbrace{\cdots}_{n} \uparrow \underbrace{\cdots}_{r} \rangle_{x=1} = \langle \underbrace{\cdots}_{l} \downarrow \underbrace{\cdots}_{n} \downarrow \underbrace{\cdots}_{r} \rangle_{x=1} \tag{4.55}$$

and

$$\langle \underbrace{\cdots}_{l} \uparrow \underbrace{\cdots}_{n} \downarrow \underbrace{\cdots}_{r} \rangle_{x=1} = \langle \underbrace{\cdots}_{l} \downarrow \underbrace{\cdots}_{n} \uparrow \underbrace{\cdots}_{r} \rangle_{x=1} \tag{4.56}$$

where l, n, $r \in \mathbb{Z}_0^+$. In the case of one spin (4.45) shows that the expectation value does not change under translation of the spin. The two spin expectation value is not translationally invariant but it has following symmetry

$$\langle \underbrace{\cdots}_{l} \uparrow \underbrace{\cdots}_{n} \downarrow \underbrace{\cdots}_{r} \rangle_{x=1} = \langle \underbrace{\cdots}_{r} \downarrow \underbrace{\cdots}_{n} \uparrow \underbrace{\cdots}_{l} \rangle_{x=1} \tag{4.57}$$

as can be seen by writing the numerator of (4.57) as

$$\begin{aligned} Z_k^{l\uparrow_n}(x)|A_1(A_0+A_1)^r &= (1+x)^{-2\beta}|(A_0+A_1)^l A_0(A_0+A_1)^n A_1(A_0+A_1)^r \\ &= \sum_{i=1}^{2^{l+n+r}} [(a+c)x+b+d]_i^{-2\beta}, \end{aligned} \tag{4.58}$$

where a, b, c, d are entries of the ith matrix from the set $(A_0+A_1)^l A_0(A_0+A_1)^n A_1(A_0+A_1)^r$. Thus for $x=1$ the sum does not change under transposition of matrices and we get (4.57).

4.5 Infinitely long spin chain

In this section, we consider expectation values and correlations for infinitely long chains. First note that by (4.45), at $x=1$, the spin expectation value has no edge effects for any finite chain. Thus, allowing $l \to \infty$ and $r \to \infty$, a spin up (down) has still probability one half. (4.50) shows that this is not true at $x=0$ (Knauf model).

In order to see the edge effect at the right side of an infinitely long chain we go back to (4.46) and let $l \to \infty$. Using (4.30) and the properties of the eigenfunction $a(x)$ we get

$$\lambda^{r+1}a(0) = 2\,a(x)|A_0(A_0+A_1)^r|_{x=0} + a(1) - a(0) \tag{4.59}$$

for all $r \geq 0$. We can write the expectation value for a spin at $r+1$ position of the infinitely long chain using (4.41)

$$\langle \underbrace{\ldots}_{\infty} \uparrow \underbrace{\ldots}_{r} \rangle_{x=0} = \frac{1}{2} + \lambda(\beta)^{-(1+r)} - \frac{\lambda(\beta)^{-r}}{2}, \tag{4.60}$$

where the eigenvalue $\lambda \in (1,2]$ for $\beta \in [0,\beta_c)$. Note that a similar expression for the spin down follows since their sum must be one. We can use the above formula at the critical temperature (when $\lambda(\beta_c)=1$) by taking the limit $\beta \to \beta_c$. Then the probability of a spin up is 1 for any finite distance r from the right (this can also be shown directly from (4.50) - showing that $K \to 1$ when $\beta \to \beta_c$ then $l \to \infty$). On the other hand for any $\beta < \beta_c$ the spin up probability goes to one half as $r \to \infty$.

Note that (4.60) also gives the right edge correlation length ξ_r and

$$\xi_r = \frac{1}{\ln \lambda} = \frac{1}{f_s}. \tag{4.61}$$

This interesting equation directly relates edge and bulk behaviour, since the bulk correlation length $\xi \propto \frac{1}{f_s}$, see Chapter 3 and (Fiala and Kleban, 2004), thus

$$\xi_r \propto \xi \propto \frac{\ln \epsilon}{\epsilon} \tag{4.62}$$

as $\beta \to \beta_c$, where $\epsilon = \frac{\beta_c}{\beta} - 1$.

Now the limit $r \to \infty$, keeping l finite. Using (4.46) we can write

$$\lim_{r\to\infty} \frac{Z_l(x)|A_0(A_0+A_1)^r}{\lambda^r} = \frac{\lambda^{l+1}}{2}\, a(0) \tag{4.63}$$

for any $\lambda > 1$. Note that $0 < a(0) < \infty$ for $\lambda \in (1, 2]$ ((4.77) below). Using (4.50) and (4.63) we obtain

$$\langle \underbrace{\cdots}_{l} \uparrow \underbrace{\cdots}_{\infty} \rangle_{x=0} = \frac{1}{2} \tag{4.64}$$

for all $l \geq 0$ and $\beta < \beta_c$. Thus the left edge effects on one spin vanish.

Next we look at the two spin case where left part of the spin chain is going to infinity. Using (4.18), (4.30) and (4.53) we get

$$\langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \uparrow \underbrace{\cdots}_{r} \rangle_x \quad = \quad \frac{a(x)|A_0(A_0+A_1)^n A_0(A_0+A_1)^r}{\lambda^{n+r+2} a(x)}. \tag{4.65}$$

It is convenient to define two functions of x and β (note that as for $a(x)$ we do not explicitly indicate the β dependence),

$$U_n(x) = a(x)|A_0(A_0+A_1)^n \tag{4.66}$$

for spin up and similarly for spin down

$$D_n(x) = a(x)|A_1(A_0+A_1)^n. \tag{4.67}$$

Clearly for all $n \geq 0$ and $0 \leq \beta < \beta_c = 1$

$$U_n(x) + D_n(x) = \lambda^{n+1} a(x) \tag{4.68}$$

and using (4.11), and (4.39)

$$U_n(x) = x^{-2\beta} D_n(1/x). \tag{4.69}$$

Note that for the chain with infinite long beginning we can write

$$\langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \rangle_x = \frac{U_n(x)}{U_n(x) + D_n(x)} \tag{4.70}$$

and for $x = 1$ it immediately follows that (since by (4.69) $U_n(1) = D_n(1)$)

$$\langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \rangle_{x=1} = \frac{1}{2} \tag{4.71}$$

as already shown (see (4.64). The result (4.60) for one spin at $x = 0$ follows from equation (4.59) which we rewrite as

$$U_n(0) = \left(\frac{1}{2}(\lambda(\beta)^{n+1} - \lambda(\beta)) + 1 \right) a(0), \tag{4.72}$$

and similarly

$$D_n(0) = \left(\frac{1}{2}(\lambda(\beta)^{n+1} + \lambda(\beta)) - 1 \right) a(0). \tag{4.73}$$

Now return to equation (4.65) for $r \to \infty$. First we write (4.65) as

$$\begin{aligned} \langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \uparrow \underbrace{\cdots}_{r} \rangle_x &= \frac{U_n(x) | A_0(A_0 + A_1)^r}{\lambda^{n+r+2} a(x)} \\ &= \frac{\sum U_n(\frac{ax+b}{cx+d})(cx+d)^{-2\beta}}{\lambda^{n+r+2} a(x)}, \end{aligned} \tag{4.74}$$

where the sum has 2^r terms with a, b, c and d from $A_0 M_r = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Note that we start with matrix A_0 and thus $\frac{ax+b}{cx+d} \leq 1$ for all $x \in \mathbb{R}_0^+$. The maximum value of $U_n(x)$ is at $x = 0$ and minimal value at $x = 1$. It follows directly from

$$U_n(x) = \lim_{k \to \infty} \sum_{i=1}^{2^{k+n}} \frac{(cx+d)_i^{-2\beta}}{\lambda(\beta)^k}, \tag{4.75}$$

where $\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \{A_0(A_0 + A_1)^k A_0(A_0 + A_1)^n\}$. Thus we can write

$$\frac{U_n(1) Z_r(x)}{\lambda^{n+r+2} a(x)} \leq \langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \uparrow \underbrace{\cdots}_{r} \rangle_x \leq \frac{U_n(0) Z_r(x)}{\lambda^{n+r+2} a(x)} \tag{4.76}$$

for all $r \geq 0$ and all $x \in \mathbb{R}_0^+$. In the limit $r \to \infty$ we get, using (4.72) and $U_n(1) = \lambda^{n+1}a(1)/2$ (see (4.68) and (4.69))

$$(\lambda - 1)\frac{a(0,\lambda)}{2\lambda} \leq \langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \uparrow \underbrace{\cdots}_{\infty} \rangle_x \leq \left(1 + \frac{2-\lambda}{\lambda^{n+1}}\right)\frac{a(0,\lambda)}{2\lambda}. \tag{4.77}$$

Note that since the correlation length $\xi = \frac{1}{\ln \lambda}$, the n-dependence of the upper bound in (4.77) is what one expects for the correlation function. We have not been able to prove this, however.

Now we will explore some edge effects (i.e. small r) for $x = 0$. When $r = 0$ we have

$$\langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \uparrow \rangle_{x=0} = \frac{U_n(0)}{\lambda^{n+2}a(0)} = \left(1 + \frac{2-\lambda}{\lambda^{n+1}}\right)\frac{1}{2\lambda}. \tag{4.78}$$

Similarly

$$\langle \underbrace{\cdots}_{\infty} \downarrow \underbrace{\cdots}_{n} \uparrow \rangle_{x=0} = \frac{D_n(0)}{\lambda^{n+2}a(0)} = \left(1 - \frac{2-\lambda}{\lambda^{n+1}}\right)\frac{1}{2\lambda}, \tag{4.79}$$

$$\langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \downarrow \rangle_{x=0} = \frac{U_n(1)}{\lambda^{n+2}a(0)} = \frac{\lambda - 1}{2\lambda}, \tag{4.80}$$

and

$$\langle \underbrace{\cdots}_{\infty} \downarrow \underbrace{\cdots}_{n} \downarrow \rangle_{x=0} = \frac{D_n(1)}{\lambda^{n+2}a(0)} = \frac{\lambda - 1}{2\lambda}. \tag{4.81}$$

Note that both (4.80) and (4.81) are completely independent of the spin separation n.

We can observe that all above is based on our knowledge of $U_n(x)$ at two points $x = 0$ and $x = 1$ (similarly for $D_n(x)$). It is easy to find generalizations of this property. We need combinations of spins for which the corresponding product of matrices A_0 and A_1 provides products with $b = 0$ or $b = 1$ and $d = 1$. This is true for chain of A_0 matrices of any length and chains starting with A_1 following by a chain of A_0 matrices of any length. These two cases give us

$$\langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \underbrace{\uparrow \cdots \uparrow \cdots \uparrow}_{r} \rangle_{x=0} = \left(1 + \frac{2-\lambda}{\lambda^{n+1}}\right)\frac{1}{2\lambda^r} \tag{4.82}$$

and

$$\langle \underbrace{\cdots}_{\infty} \uparrow \underbrace{\cdots}_{n} \downarrow \underbrace{\uparrow \cdots \uparrow \cdots \uparrow}_{r} \rangle_{x=0} \quad = \quad \frac{\lambda - 1}{2\lambda^{r+1}}. \tag{4.83}$$

Similarly we get

$$\langle \underbrace{\cdots}_{\infty} \downarrow \underbrace{\cdots}_{n} \underbrace{\uparrow \cdots \uparrow \cdots \uparrow}_{r} \rangle_{x=0} \quad = \quad \left(1 - \frac{2-\lambda}{\lambda^{n+1}}\right) \frac{1}{2\lambda^{r}} \tag{4.84}$$

and

$$\langle \underbrace{\cdots}_{\infty} \downarrow \underbrace{\cdots}_{n} \downarrow \underbrace{\uparrow \cdots \uparrow \cdots \uparrow}_{r} \rangle_{x=0} \quad = \quad \frac{\lambda - 1}{2\lambda^{r+1}}. \tag{4.85}$$

Note that (4.83) and (4.85) are independent of n. Also note that to calculate any of (4.78) - (4.85) could require knowing the four values $U_n(1/2)$, $U_n(2)$, $D_n(1/2)$ and $D_n(2)$. Using (4.69) accounts for two of these, in addition (4.68) removes one more, but are left with one unknown value. For general x, one has four unknown quantities.

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

In this thesis, we have examined the thermodynamics of several statistical models defined on the Farey fractions. There are several main results. In Chapter 2, we introduce several models, and that they all have the same free energy. This means, in particular, that they all have a single phase transition, which is (barely) second-order. The asymptotic behavior of the free energy at the transition is determined by making use of some results of Prellberg (Prellberg and Slawny, 1992; Prellberg, 1991). In Chapter 3, we extend the spin chain models by introducing a magnetic field. Using both rigorous and non-rigorous methods, we determine their phase diagram. Finally, Chapter 4 introduces a partition function that extends the "number-theoretical" spin chain of Knauf (Knauf, 1993, 1998). This allows us to establish a connection with the operator studied by Prellberg (in the context of dynamical systems) in a new and simple way. We prove that this operator is the transfer operator for these extended partition functions. Finally, the recurrence relations satisfied by the extended partition function allow us to calculate certain spin expectation values and correlation functions for the "number-theoretical" (Knauf) spin chain.

In the future, we plan to complete work with T. Prellberg on a calculation showing that a certain cluster approximation for the transfer operator $\mathcal{K}_\beta$ of Chapter 4, leads to the same phase diagram as we obtained (Chapter 3) using renormalization group arguments. In addition, we plan to make use of the methods of Chapter 4 to examine the behaviour of the function $a(x)$ at the phase transition. At $x = 0$ the limit of

(4.30) diverges, which is consistent with the divergence of $\varphi(x) = \frac{1}{x}$. This function is an eigenfunction of $\mathcal{K}_{\beta=1}$ with eigenvalue 1 if we consider the appropriate Hilbert space (Mayer and Roepstorff, 1987; Mayer, 1990). Note that $\frac{1}{x}$ is not normalizable in the Hilbert space of Prellberg (Prellberg, 2003) and corresponds to the l^∞ eigenvector of Contucci and Knauf operator. We believe that with proper normalization of (4.30) it should be possible to prove that for $x > 0$ we obtain the function $\frac{1}{x}$. This would imply that the "number-theoretical" spin chain models, at the critical temperature, are in a completely ordered state (all spins up or down). It would also imply that

$$\lim_{k\to\infty} \frac{(1+n)}{Z_k(1,1)} \sum_{i=1}^{2^{k-1}} \frac{c_i^n + d_i^n}{(c_i + d_i)^{2+n}} = 1 \tag{5.1}$$

for all $n \geq 0$, where c_i and d_i are neighbour denominators of Farey fractions in the level k. Note that $Z_k(1,1) = \sum_{i=1}^{2^k} \frac{1}{(c+d)_i^2}$, thus for $n = 0$ the above sum is one for all $k > 0$. Let us finish with two observations we have made when we analysed the conjecture (5.1). First, for every $x > 0$ and $k > 0$

$$\sum_{i=1}^{2^k} \left(\frac{1}{(cx+d)_i (ax+b)_i} + \frac{1}{(c+dx)_i (a+bx)_i} \right) = \frac{1}{x}, \tag{5.2}$$

where $\frac{a_i}{c_i}$ and $\frac{b_i}{d_i}$ are the neighbour Farey fractions at the level k. The second observation is that there are solutions of the functional equation

$$g(x,y) + g(1/x, 1/y) = 1, \tag{5.3}$$

which are generated by the action of the operator $\mathcal{K}_{\beta=1}^k$ on $(x+y)^{-1}$. Thus for every $k \geq 0$

$$g_k(x,y) := \mathcal{K}_{\beta=1}^k (x+y)^{-1},$$

where the variable y is treated as a parameter, is a solution of the functional equation (5.3). For example, the first two solutions are $g_0(x,y) = \frac{x}{x+y}$, $g_1(x,y) = x\left(\frac{1}{1+x+y} + \frac{1}{(1+x)(x+xy+y)}\right)$.

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APPENDIX

BOUNDS FOR $\frac{Z_{N+1}}{Z_N}$

We write the partition function (3.3) restricted to chains starting with A (see Section 2.1)

$$Z_N^A(\beta,h)=\sum_{n=1}^{2^N}\frac{e^{-\beta h\left(2\sum_{i=1}^N\sigma_i-N\right)}}{(d_N^{(n)}+n_N^{(n+1)})^\beta},\quad \beta\in\mathbb{R}. \tag{A.1}$$

Note that the partition function (3.3) is the sum of $Z_N^A(\beta,h)$ and $Z_N^B(\beta,h)$, where the $Z_N^B(\beta,h)$ is the partition function for chains starting with the matrix B. First we find bounds for $Z_N^A(\beta,h)$ and then prove a lemma which lets us apply the bounds for $Z_N^A(\beta,h)$ to $Z_N^B(\beta,h)$ also.

Now, when we go from level N to level $N+1$ we double the number of the terms in the partition function. Note that for chains starting with the matrix A one half of the terms come from matrix products of the form $AM_{N-1}A$ and the others from products $AM_{N-1}B$. It is easy to check that the corresponding traces for given $n\in\{1,\dots,2^N\}$ are $d_{N+1}^{(2n-1)}+n_{N+1}^{(2n)}$ and $d_{N+1}^{(2n)}+n_{N+1}^{(2n+1)}$, respectively. These traces are multiplied by an h dependent factor $e^{-\beta h\left(2\sum_{i=1}^{N+1}\sigma_i-N-1\right)}$ which is simply $e^{\beta h}$ raised to the power $(\#A-\#B)$, the number of matrices A minus the number of matices B in the particular chain. For the terms from products of the form $AM_{N-1}A$, it follows on using the definition of the Farey fractions that

$$\frac{e^{-\beta h\left(2\sum_{i=1}^{N+1}\sigma_i-N-1\right)}}{(d_{N+1}^{(2n-1)}+n_{N+1}^{(2n)})^\beta}=\frac{e^{-\beta h\left(2\sum_{i=1}^{N}\sigma_i-N\right)+\beta h}}{(d_N^{(n)}+n_N^{(n)}+n_N^{(n+1)})^\beta}\le\frac{e^{-\beta h\left(2\sum_{i=1}^{N}\sigma_i-N\right)}}{(d_N^{(n)}+n_N^{(n+1)})^\beta}e^{\beta|h|}$$

and, similarly, for $AM_{N-1}B$

$$\frac{e^{-\beta h\left(2\sum_{i=1}^{N+1}\sigma_i-N-1\right)}}{(d_{N+1}^{(2n)}+n_{N+1}^{(2n+1)})^\beta}=\frac{e^{-\beta h\left(2\sum_{i=1}^{N}\sigma_i-N\right)-\beta h}}{(d_N^{(n)}+d_N^{(n+1)}+n_N^{(n+1)})^\beta}\le\frac{e^{-\beta h\left(2\sum_{i=1}^{N}\sigma_i-N\right)}}{(d_N^{(n)}+n_N^{(n+1)})^\beta}e^{\beta|h|}.$$

For the lower bound we just need the $AM_{N-1}A$ terms

$$\frac{e^{-\beta h\left(2\sum_{i=1}^{N}\sigma_i - N\right)+\beta h}}{(d_N^{(n)} + n_N^{(n)} + n_N^{(n+1)})^\beta} \geq \frac{e^{-\beta h\left(2\sum_{i=1}^{N}\sigma_i - N\right)}}{(d_N^{(n)} + n_N^{(n+1)})^\beta}\frac{e^{-\beta|h|}}{2^\beta},$$

where we used the fact that $n_N^{(n)} \leq d_N^{(n)}$. Thus we get for $Z_{N+1}^A(\beta,h) = Z_{N+1}^{AM_{N-1}A}(\beta,h) + Z_{N+1}^{AM_{N-1}B}(\beta,h)$

$$2^{-\beta}e^{-\beta|h|}Z_N^A(\beta,h) \leq Z_{N+1}^A(\beta,h) \leq 2e^{\beta|h|}Z_N^A(\beta,h)$$

for any $\beta \geq 0$ and $h \in \mathbb{R}$.

Finally, we prove a lemma which allows us to bound $Z_N^B(\beta,h)$. Consider a (2×2) matrix $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ and define the operator $\sim$ via $\tilde{M} := \begin{pmatrix} m_4 & m_3 \\ m_2 & m_1 \end{pmatrix}$. Then we have the following result.

Lemma .1 *Let $M = AZ_1Z_2\ldots Z_N$, where $Z_i \in \{A,B\}$, with $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\tilde{M} = B\tilde{Z}_1\tilde{Z}_2\ldots\tilde{Z}_N$, i.e. the $\sim$ operator exchanges A and B.*

Proof. We will use mathematical induction. It is easy to see that $A = \tilde{B}$ and $B = \tilde{A}$. From matrix multiplication follows $B\tilde{M} = \begin{pmatrix} m_2+m_4 & m_1+m_3 \\ m_2 & m_1 \end{pmatrix}$ and $AM = \begin{pmatrix} m_1 & m_2 \\ m_1+m_3 & m_2+m_4 \end{pmatrix}$.

Clearly the $\sim$ operation is a 1-to-1 map of the set of all chains AM_N onto BM_N. Furthermore, the magnetic field term in the energy of each chain changes sign under this operation, so that the bounds just obtained for $Z_N^A(\beta,h)$ may be applied to $Z_N^B(\beta,h)$. Therefore

$$2^{-\beta}e^{-\beta|h|} \leq \frac{Z_{N+1}}{Z_N} \leq 2e^{\beta|h|}.$$

Note that the proof is easily adapted to the KSC model.

BIOGRAPHY OF THE AUTHOR

Jan Fiala was born in Prague, Czech Republic on March 3, 1974. He graduated from Stepanska Academic High School in 1992. He was a member of the Czech Republic Yacht team from 1983 to 1989. He became champion of Czech Republic in 1989 and represented his country in the world championship in the Netherlands the same year. He also holds a silver medal from the Czech championship in 1988, bronze from the Czechoslovak championship 1989, and from international competitions a gold in Poland (1985) and a silver in Hungary (1988). He represented his high school in the Czech Physics Olympics (6th place in Prague 1992).

Jan was accepted in 1992 as a student in the Czech Technical University, Prague, in the Nuclear Sciences and Physical Engineering faculty. During his studies he worked as a research assistant in the Czech Academy of Science on his Master's degree project. He graduated in Mathematical Engineering in 1997 and continued in the same department as a Ph.D. student. Jan came to the University of Maine Department of Mathematics as a transient student for the spring semester of 1998. During the academic year 1998 - 99 he continued his graduate studies in the Czech Republic and taught physics at Jaroslav Heyrovsky High school. He came back to the University of Maine Department of Physics as a graduate student in 1999. The contents of Chapter 2 and 3 have been published in The Journal of Statistical Physics: Fiala, J., Kleban, P., and Ozluk , A. (2003), "The phase transition in statistical models defined on Farey fractions", *J. Stat. Phys.*, **110**: 73-86 and Fiala, J. and Kleban, P. (2004), "Thermodynamics of the Farey Fraction Spin Chain", *J. Stat. Phys.*, **116**: 1471-1490, respectively.

Jan practiced martial arts for 14 years (Karate, Kungfu, Musado), enjoys tennis, racquetball, volleyball (was a member of University Maine Volleyball Club), table tennis (second place in Maine campus competition). In Maine he became interested in photographing nature. He received the best picture award at the Bangor Garden Show in 2003 and 2004. Several of his pictures were published by the International Library of Photography and Bangor Daily News.

Jan is a candidate for the Doctor of Philosophy degree in Physics from The University of Maine in December, 2004.